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This study generalizes the assumption of perfect capital markets as one cornerstone assumption in the classical Faustmann forest rotation model. We show that an infinite chain of optimal rotations is determined by a nonlinear difference equation with a saddle point property. The rotation period is nonconstant but approaches a stationary rotation in finite time. The length of optimal rotation, timber supply and the value of forest land depend on forest owner specific parameters that are normally nonexistent in the rotation problems. The model explains several empirical findings that cause problems to the classical versions of the Faustmann model.

Klassinen optimikiertoaikamalli perustuu täydellisiin pääomamarkkinoihin eli oletuksiin, että metsänomistaja voi saada lainaa ja sijoittaa rahaa samalla korkokannalla ilman rajoitteita. Nämä oletukset eivät kuitenkaan tyypillisesti esimerkiksi Suomessa ole voimassa. Tämä on seurausta siitä, että lainasta maksettu korko on useimmiten suurempi kuin metsänomistajan sijoituksilleen saama korko, metsämaa puustoineen ei välttämättä kelpaa lainan vakuudeksi koko rahallisesta arvostaan ja metsänomistaja voi eri syistä olla luotonsaantirajoitusten alaisena.

Vaikka pääomamarkkinoiden epätäydellisyyksiä voidaan pitää tyypillisinä, ei niiden vaikutuksista optimikiertoajan määräytymiseen, metsämaan arvoon ja metsänhoitotoimenpiteiden taloudelliseen kannattavuuteen juurikaan ole kansainvälistä metsäekonomista tutkimustietoa. Aijemmin julkaistu epätäydellisiä pääomamarkkinoita ja puuntarjontaa koskeva tutkimus ei perustu kiertoaikamalliin vaan sensijaan metsän kuvaukseen homogeenisena biomassana ilman ikäluokkia. Homogeenisen biomassan korjuuta kuvaavat metsämallit eivät tuota tuloksia kiertoajasta, maan arvosta tai taloudellisista kannustimista istuttaa uusi puusto päätehakatun tilalle. Tämän seurauksena "perhemetsätalouden" ymmärrys ja metsäneuvonnan tutkimuksellinen perusta ovat puutteellisia.

Epätäydellisten pääomamarkkinoiden sisällyttämisellä optimikiertoaikamalliin on selvää empiiristä ja käytännön merkitystä. Metsänomistajille tehdyssä opaskirjassa (Mielikäinen, K. ja Riikilä, M. 1997, *Kannattava Puuntuotanto*, Metsäntutkimuslaitos ja Tapio Metsälehtikustannus, s. 15, 30) esitetään metsänomistajien metsäpääomalleen asettamien tuottovaatimusten riippuvan siitä onko metsänomistaja "säästäjä", "käyttäjä" vai "sijoittaja". Esitetty kuvaus ei voi päteä täydellisillä pääomamarkkinoilla, koska näillä toimiva rationaalinen metsänomistaja ottaa markkinakoron annettuna riippumatta säästämistä/kuluttamista koskevista tavoitteistaan. Jos kysymyksessä on epätäydellisten pääomamarkkinoiden olosuhteisiin tarkoitettu neuvonta, puuttuu esitetyiltä ajatuksilta tutkimuksellinen pohja.Tätä pyritään tässä tutkimuksessa kehittämään.

Klassisen kiertoaikamallin ongelmana on, ettei se selitä miksi esimerkiksi Suomessa ja Yhdysvalloissa yksityismetsänomistajien puuntarjontaa selittävät metsänomistajakohtaiset tekijät kuten metsätalouden ulkopuoliset tulot, varallisuus tai ikä. Se ei selitä miksi metsätulojen realisoitumisen ja kulutuksen välillä voidaan havaita empiirinen yhteys (ns. "volvo argumentti"). Lisäksi klassinen kiertoaikamalli näyttää epäonnistuvan metsämaan markkinahinnan ennustamisessa ja selittämisessä. Tällä on myös Suomessa kasvava merkitys metsämaan kaupan vapauduttua ja esimerkiksi laskettaessa korvauksia lunastettaessa metsämaata luonnonsuojelutarkoituksiin.

Suomen metsäpolitiikan suunnittelussa paljon käytetty MELA -malli nojaa täydellisten pääomamarkkinoiden oletukseen ja klassiseen kiertoaikamalliin. MELA-mallin kehitystyön yhteydessä tehty yritys käsitellä tapausta, jossa anto- ja ottolainakorot eroavat on kiinnostava, mutta on metsänomistajan preferenssejä koskevan kuvauksen osalta hyvin rajoittunut (Lappi ja Siitonen 1985).

Tässä tutkimuksessa pääomamarkkinoiden epätäydellisyys muotoillaan lainarajoitteen muodossa. Tämä kuvaa myös tilannetta, jossa sijoituksia ja lainaa koskevat korot eroavat toisistaan olettaen, että lainakorko on riittävän suuri aiheuttamaan metsänomistajan pidättäytymisen lainanotosta. Lainarajoite sisällytetään malliin, joka yhdistää klassisen optimikiertoaikamallin taloustieteessä vakiintuneeseen mallin kotitalouden säästämistä/kuluttamista koskevasta päätöksenteosta.

Osoittautuu, että epätäydellisillä pääomamarkkinoilla optimirotaatio ei ole vakio kuten klassisessa kiertoaikamallissa. Ajan kuluessa kiertoaika kuitenkin lähestyy vakiopituista, stationääristä kiertoaikaa, joka on varallisuuttaan vähentävälle metsänomistajalle klassista optimikiertoaikaa lyhempi. Vakiona toistuva stationäärinen kiertoaika riippuu negatiivisesti puun hinnasta ja positiivisesti uuden puuston perustamiskustannuksista kuten klassinen kiertoaika, mutta päinvastoin kuin klassinen kiertoaika, on sitä pidempi mitä suurempi on markkinakorko. Lisäksi stationäärinen kiertoaika on sitä pidempi mitä suuremmat ovat metsätalouden ulkopuoliset tulot ja mitä alhaisempi on metsänomistajan subjektiivinen korkokanta. Näinollen malli selittää esimerkiksi sen miksi lyhyen aikavälin hyvinvointia painottava metsänomistaja on taipuvainen päätehakkaamaan metsänsä niin varhaisessa kasvuvaiheessa kuin laki sallii. Klassisen kiertoaikamallin mukaan "lyhytnäköinen" metsänomistaja soveltaa samaa kiertoaikaa kuin pitkän aikavälin hyvinvointia painottava metsänomistaja.

Suomessa päätehakattavan puuston täytyy toteuttaa joko minimi-ikä- tai minimijäreyskriteeri. Tämän tutkimuksen perusteella voi otaksua, että lainarajoitteisen metsänomistajan näkökulmasta tämä säädös saattaa olla taloudellisesti optimaalista hakkuuajankohtaa rajoittava tekijä.

Metsänomistajille suunnatussa neuvonnassa (vrt. Mielikäinen ja Riikilä 1997) näyttäisi olevan perusteltua jäsentää päätöksentekotilannetta sen mukaan onko metsänomistaja velaton ja omaisuuttaan kasvattava vai velkaantunut ja mahdollisesti lainarajoitteinen. Ensimmäisessä vaihtoehdossa kiertoaika määräytyy klassisen kiertoaikamallin mukaan ja korkona käytetään sijoituksille/säästöille saatavaa (verojenjälkeistä) korkokantaa. Toisessa vaihtoehdossa sovelletaan veloista maksettavaa (verojenjälkeistä) korkoa, mutta jos metsänomistaja on lisäksi lainarajoitteinen voi optimikiertoaika olla velkakorolla laskettua klassisen kiertoaikamallin suositusta oleellisesti lyhempikin.

Klassinen kiertoaikamalli antaa teoreettisen taustan esimerkiksi Suomessa sovellettavalle metsämaan arvon laskennalle. Tässä tutkimuksessa tehty numeerinen tarkastelu osoittaa, että pääomamarkkinoiden epätäydellisyyksien seurauksena metsämaan arvo voi olla oleellisesti klassisen mallin arviota alhaisempi. Paljaan maan todellinen arvo metsänomistajalle voi olla negatiivinen, vaikka klassisen malli ennustaa positiivista maan arvoa. Toisin sanoen, pääomamarkkinoiden epätäydellisyyksien seurauksena kannustimet istuttaa uusi metsä päätehakatun tilalle voivat puuttua, vaikka klassisen mallin perusteella investointi näyttää kannattavalta. Tämä osoittaa, että pääomamarkkinoiden toimivuus ja metsänomistajan varallisuuteen nähden rajoittamaton lainansaanti ovat kestävän metsätalouden perusedellytys, joiden puuttumista on Suomessa jouduttu paikkaamaan mm. lainsäädännöllisin toimenpitein.

Key words: forest rotation model, Faustmann model, timber supply, value of forest land

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Content

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1 Introduction

Based on 150 years of history, the forest rotation model by Faustmann (1849), Pressler (1860) and Ohlin (1921) possesses several unique features as a description of intertemporal decision making. Under perfect markets and perfect foresight, the model specifies the rotation period for an even aged stand that maximizes the present value of forest land over an infinitely long time horizon (Samuelson 1976). The ongoing research has yielded new extensions with various kinds of uncertainties (Miller and Voltaire 1983, Clarke and Reed 1989, 1990, Willassen 1998), nonlinearities (Heaps 1984, Mitra and Wan 1986) and models for studying the utilization of the global forest resources (Lyon and Sedjo 1990). The forest rotation model also has high practical relevance. Biologically oriented forest scientists have developed highly detailed numerical procedures for determining optimal forest management procedures (e.g. Haight 1985). In the Nordic countries detailed guidance on how to obtain field estimates for the financial maturity of a growing forest stand are based on the Faustmann model as well as the calculations for the value of forest land.[1]

In spite of its unique success the Faustmann model faces difficulties with empirical observations on nonindustrial private forest owners in Northern Europe and various parts of North America. A number of econometric investigations have revealed that forest owners' harvesting decisions depend on owner-specific characteristics like nonforest income, wealth and owner's age, which are excluded from the classical Faustmann model (e.g. Binkley 1981, Romm et al. 1987, Dennis 1988, 1989, Jammick & Becket 1987, Kuuluvainen and Salo 1991). There is also some empirical evidence on the dependence of forest land prices on owner characteristics (Aronsson and Carlén 1997). In the Nordic forestry folklore the "volvo argument" states that harvests can be explained by the forest owner's need for a new

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[1]As discussed by Miller and Voltaire (1983), problems similar to the rotation model occur in various other fields in economics as well. For example, some optimal stopping problems come close to the rotation model.

vehicle (Johansson and Löfgren 1985, p. 138, Malmborg 1967). Forest experts believe that forest owners have an incentive to use a forest as a "bank" for direct financing of their consumption expenditures. These findings are disturbing for the Faustmann approach based on perfect capital markets because the model, by assuming profit maximization, implicitly relies on the Fisherian separation theorem and predicts that forest owner-specific factors should play no role in explaining cutting decisions. In addition, there should be no temporal connection between harvesting and peaks in consumption levels.

This study explains these findings by relaxing one basic assumption in the Faustmann tradition, i.e. the assumption of perfect capital markets. In his widely cited paper, Samuelson (1976) indicates this as one important future extension of the rotation model. However, the progress in this line of research has been slow or nonexistent. While relaxing the perfect capital market assumption, a number of studies have given up the age-class structure and even aged harvesting approach inherent in the Faustmann rotation model (e.g. Johansson and Löfgren 1985, p. 8, Koskela 1989, Kuuluvainen 1990). Instead, these models describe forests as homogeneous biomass. This choice increases analytical tractability but undoubtedly the original flavor of the Faustmann model and its strong biological base are lost. Murphy et al. (1977) study the effects of imperfect capital markets in the original rotation framework. They search for numerical solutions using dynamic programming but obtain solutions labelled as 'globally nonoptimal', due to the artificial dependence of the length of the planning horizon. Another study (Lappi and Siitonen 1985) assumes a divergence between lending and borrowing rates of interest and applies linear programming. Postulating linear utility from consumption and a max-min type intertemporal objective function, consumption (or utility) is constant over time. Their numerical examples suggest that imperfect capital markets may provide an interpretation for the smoothness requirements for forest income, an idea foresters normally apply ad hoc.

We formulate a continuous-time utility maximization consumption/savings life cycle model augmented by forest capital within the Faustmann rotation framework. Imperfections in the capital markets are specified as a constraint on borrowing. This may reflect credit rationing or the fact that in many countries standing forests are not accepted as securities to their full financial (or Faustmann) value. In addition, a constraint on borrowing must partially reveal the more complex case of different lending and borrowing rates of interest since a sufficiently high borrowing rate of interest is equivalent to a borrowing constraint. We assume an infinite time horizon, which reflect a perfectly altruistic bequest motive or a "forest dynasty".

As in earlier Faustmann extensions (e.g Heaps 1984, Willassen 1998), our borrowing constraint leads to a complex optimal control problem. The model includes a pure state constraint and endogenous jump discontinuities in the two state variables. We handle the discontinuities by formulating the model as an infinite chain of control problems where jumps occur between the successive subproblems. The continuity condition for the Hamiltonian yields a nonlinear "rotation difference equation" that determines the chain for optimal harvesting dates.

If the borrowing constraint is not binding, the Faustmann solution satisfies our rotation difference equation as a stationary solution. With a binding borrowing constraint the difference equation has a stationary solution that depends on forest owner-specific characteristics. Only with specific initial financial assets and stand age does the optimal solution follow the stationary path without transitional dynamics. Along the stationary solution, consumption jumps up at the harvesting dates, contrary to the continuous consumption under perfect capital market. Assuming a specific form of the utility function, we obtain a rather complete set of analytical results for the comparative statics of the stationary rotation. To derive these results, we extend a lemma by Pontryagin et al. (1962, p.122). It is shown that, for example, nonforest income increases and the borrowing constraint decreases the rotation length. In contrast to the Faustmann model, the rotation length is longer, the higher the rate of interest. Assuming the stationary solution, we show by numerical computation that the borrowing constraint causes major changes in the determination of the value of forest land and incentives for planting a new stand after harvest.

Linearization and numerical analysis of the third order rotation difference equation reveals that it possesses the saddle point property. With a given level of initial nonforest assets and stand age, the optimal rotation length converges toward the stationary solution. For forest owners with saving incentives, the borrowing constraint may become ineffective after e.g. one harvest. In such a case the Faustmann solution is reached in finite time. A brief review of econometric timber supply studies suggests that the empirical results causing difficulties for the classical Faustmann model may be explained by our extended model.

All these results are new in the forest economic literature. A preliminary analysis in Tahvonen (1997) uses a similar formulation but is restricted to the stationary solution and cases with zero rate of interest and nonforest income. Tahvonen and Salo (1997) show that (under perfect capital markets) also the *in situ* or amenity preferences lead to nonstationary rotation programs and violation of the Fisherian separation theorem. However, the context of that study and the specific dynamic features of the model are clearly different that those studied here.

The paper is organized as follows. Section 2 develops the model and shows how the Faustmann solution follows as a special case. Section 3 focuses on stationary, and section 4 on nonstationary, solutions. Section 5 includes empirical remarks and suggests extensions.

2 The forest owner's decision problem

The forest owner's problem is to choose his consumption time path c and the cutting moments t_i so as to maximize his life cycle welfare. As in any economic description of life cycle decision making, it is not possible to circumvent the question of bequest motives. Theoretically, an important benchmark is obtained by assuming that each generation of forest owners takes the welfare of future generations as if they themselves lived infinitely (see e.g. Ihori 1996). This benchmark is useful here since it enables us to study the effects of the borrowing constraint without additional complications to the model.

U(c) denotes a strictly concave utility function with $U'(c)\to\infty$ as $c\to 0$, where c is consumption. The rate of subjective time preference is δ and the rate of interest ρ. The level of financial assets is a and the constant level of exogenous nonforestry income m. The commercial stand volume x is a function of stand age s. There exist $0<\underline{s}<\bar{s}$ such that for all $s\in(\underline{s},\bar{s})$ it holds that $x(s)>0$ and, as $s\to\bar{s}$, $\dot{x}=F[x(s)]\to 0$. In addition, it holds that $F(x)>0$ and $F''(x)<0$. The logistic growth[2] function satisfies these requirements in the form $\underline{s}=0$ and $\bar{s}=\infty$. The variable t denotes calendar time and t_i the harvesting moments. The left hand limit of t at t_i is t_i^-. When $t\in[t_{i-1}, t_i)$, the stand age along the ith rotation is $s_i=t-t_{i-1}$. The (constant) stumpage price and regeneration costs are denoted by p and w respectively.

The forest owner's problem with borrowing constraint is to

$$\max_{\{c,t_1,t_2,...\}} \quad W=\int_0^\infty U(c)e^{-\delta t}dt, \tag{1}$$

$$\text{s.t.} \quad \dot{a}=\rho a-c+m, \text{ when } t\notin t_i,\ i=1,...,\infty, \tag{2}$$

$$a(t_i)=a(t_i^-)+px(t_i^- - t_{i-1})-w,\ i=1,...,\infty, \tag{3}$$

$$a(t)\geq 0, \tag{4}$$

$$a(0)=a_0, \tag{5}$$

$$\dot{x}=F(x), \text{ when } t\notin t_i,\ i=1,...,\infty, \tag{6}$$

$$x(t_i)=x_0,\ i=1,...,\infty, \tag{7}$$

$$x(0)=\underline{x}_0, \tag{8}$$

where $t_0=0$ and a_0, $\underline{x}_0$ denote the initial levels of financial assets and forest stand

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[2]In this case $x(s)=K/(1-Ce^{-rs})$, where $C=(1-K/x_0)$, where x_0,K,r are positive constants. Another growth function that we will use in numerical examples is $x(s)=a-be^{-cs}$, when $a-be^{-cs}>0$ and $x(s)=0$ otherwise.

respectively. In (7) the volume of forest stand immediately after the regeneration is x_0. Equation (3) shows the level of financial assets after the harvest. Restriction (4) is the borrowing constraint, which, for simplicity, restricts financial assets to be nonnegative.

The complexity of this problem follows from the fact that stand volume and financial assets jump discontinuously at the harvesting moments. These jumps and the related necessary conditions can be studied by specifying the problem in the form:

$$W_i(a_{i-1},t_{i-1})=\max_{\{t_i,c\}} \left\{\int_{t_{i-1}}^{t_i} U(c)e^{-\delta t}dt+W_{i+1}\left[a(t_i^-)+px(t_i-t_{i-1})-w,t_i\right]\right\}, \tag{9}$$

s.t. $\quad a(t_{i-1})=a_{i-1},$ (10)

and (2), (4), (6)-(8) and i=1,...,∞. In (9), W_{i+1} is the value function for the rotation periods starting at i+1 and it is defined by a problem analogous to the maximization of W_i. In (10) a_{i-1} denotes the initial asset level at t_{i-1}. When $t_i=t_0$, write $t_0=0$, $a(0)=a_0$ and $x(0)=\underline{x}_0$. Due to the pure state constraint, $a(t)\geq 0$, we apply a scrap value extension of theorem 2, p. 332 in Seierstad and Sydsæter (1987). The Hamiltonian for any subperiod i is given by $H_i=U(c)e^{-\delta t}+\mu(\rho a+m-c)+\varphi F(x)$ and the Lagrangian by $L_i=H_i-\tau(\rho a+m-c)$, where μ and φ are the present value costates for financial assets and stand volume respectively. The shadow price τ is associated with the constraint $\dot{a}(t)\geq 0$ when a=0. We obtain the following necessary conditions:

$U'(c)e^{-\delta t}-\mu=0,$ (11)

τ is constant on any interval where a>0, (12i)

τ is continuous at all $t\in(t_{i-1},t_i)$ where a=0 and $\dot{a}$ is discontinuous, (12ii)

$\mu*=\mu+\tau,$ (13i)

where $\mu*$ is continuous, and $\mu*$ has a continuous derivative $\dot{\mu}*$

at all points of continuity of c and τ, (13ii)

$$\dot{\mu}^*=-\partial L/\partial a=-\rho\mu^*+\tau\rho, \tag{14}$$

$$\dot{\varphi}=-\varphi F'(x), \tag{15}$$

$$\mu(t_i^-)-\partial W_{i+1}/\partial a(t_i^-)\geq 0,\ [\mu(t_i^-)-\partial W_{i+1}/\partial a(t_i^-)]\,a(t_i^-)=0,\ a(t_i^-)\geq 0, \tag{16}$$

$$\varphi(t_i^-)=\partial W_{i+1}/\partial x(s_i), \tag{17}$$

$$H_i(t_i)=-\partial W_{i+1}/\partial t_i, \tag{18}$$

and where μ has one-sided limits everywhere and τ is nondecreasing. In (16), $\partial W_{i+1}/\partial a(t_i^-)=\mu(t_i)$ and, in (18), $-\partial W_{i+1}/\partial t_i=H_{i+1}(t_i)$ (see Seierstad and Sydsæter 1987, theorem 11, p. 215). Thus (18) yields

$$U[c(t_i^-)]e^{-\delta t_i}+\mu(t_i^-)[\rho a(t_i^-)+m-c(t_i^-)]+\varphi(t_i^-)F[x(s_i)]=$$
$$U[c(t_i)]e^{-\delta t_i}+\mu(t_i)\{\rho[a(t_i^-)+px(s_i)-w]+m-c(t_i)\}+\varphi(t_i)F(x_0), \tag{19}$$

where $s_i=t_i-t_{i-1}$ is the length of the rotation period i. Equation (19) implies the continuity of the Hamiltonian function. This requirement forms the basis for studying the optimal rotation period. By (17), $\varphi(t_i^-)=p\partial W_{i+1}/\partial a=p\mu(t_i)$ and similarly $\varphi(t_{i+1}^-)=p\mu(t_{i+1})$. To develop term $\varphi(t_i)F(x_0)$ in (19), note that (15) together with (6) is a linear first order differential equation for φ. We can write its solution as

$$\varphi(t_i)=\varphi(t_{i+1}^-)e^{\int_{t_i}^{t_{i+1}^-}F'[x(t-t_i)]dt}. \tag{20}$$

Using the change of variables dt=dx/F(x)

$$\int_{t_i}^{t_{i+1}^-}F'[x(t-t_i)]dt=\int_{x(t_i-t_i)}^{x(t_{i+1}^- -t_i)}F'(x)/F(x)dx=\ln\{F[x(t_{i+1}^- -t_i)]/F[x(t_i-t_i)]\}. \tag{21}$$

Since $x(t_i-t_i)=x_0$ and $t_{i+1}-t_i=s_{i+1}$, it follows by (20), (21) and (17) that

$$\varphi(t_i)F(x_0)=\varphi(t_{i+1}^-)F[x(s_{i+1})]=p\mu(t_{i+1})F[x(s_{i+1})]. \tag{22}$$

By (16), $\mu(t_i^-)a(t_i^-)=\mu(t_i)a(t_i^-)$, which allows us to eliminate $\rho a(t_i^-)$ from both sides of (19). The Hamiltonian continuity condition obtains the form

$$U[c(t_i^-)]e^{-\delta t_i}+\mu(t_i^-)[m-c(t_i^-)]+p\mu(t_i)F[x(s_i)]=$$
$$U[c(t_i)]e^{-\delta t_i}+\mu(t_i)\{\rho[px(s_i)-w]+m-c(t_i)\}+p\mu(t_{i+1})F[x(s_{i+1})]. \tag{23}$$

This is a nonlinear difference equation. We will call it the rotation difference equation.

Faustmann solution as a special case

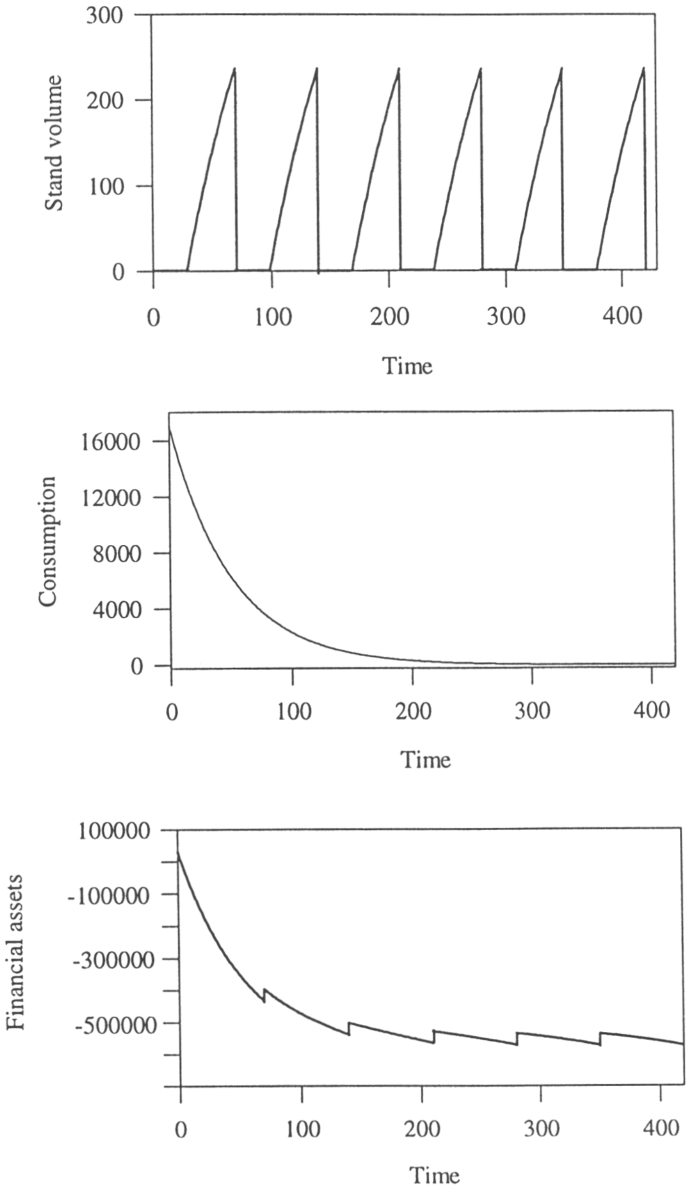
We show that (23) yields the Faustmann formula as a special case when the borrowing constraint is not binding. This analysis will help us in studying (23) in the more complex cases with the binding borrowing constraint. Without the constraint $a\geq 0$, we can assume $\tau=0$. Hence $\mu^*=\mu$ and (16) takes the form $\mu-\partial W_{i+1}/\partial a(t_i^-)=0$, implying that μ must be continuous, i.e. $\mu(t_i^-)=\mu(t_i)$ and $c(t_i^-)=c(t_i)$. By (14) $\mu(t_i)=\mu_0 e^{-\rho t_i}$ and $\mu(t_{i+1})=\mu_0 e^{-\rho t_i}e^{-\rho s_{i+1}}$. Now (23) reduces to

$$pF[x(s_i)]-\rho[px(s_i)-w]-pe^{-\rho(s_{i+1})}F[x(s_{i+1})]=0. \tag{24}$$

This is a first order nonlinear difference equation for s_i. In (24) the marginal value growth for rotation i, $pF[x(s_i)]$, must equal the interest costs $\rho[px(s_i)-w]$ for the next harvesting net income, plus the present value loss $pe^{-\rho s_{i+1}}F[x(s_{i+1})]$ which occurs if the next rotation i+1 is marginally shorter. This relationship must hold between all successive rotations up to infinity. It is possible to solve $pF[x(s_{i+1})]$ from the equation for the next harvest at t_{i+1} and then repeat this up to infinity, yielding $pF[x(s_i)]=\sum_{j=0}^{\infty}\rho[px(s_{i+j})-w]e^{-\rho(t_{i+j}-t_i)}$. Thus an equivalent form of equation (24) states that marginal value growth equals the interest costs of postponing all the coming harvests marginally, whatever the lengths of the different rotation periods. Usually, in deriving the Faustmann rotation, it is postulated a priori that all rotation periods must have equal length since, after each harvest, the problem is always the same (Faustmann 1849, Clark 1990, p. 269). In our model, however, financial assets evolve in time. Within this model we may note that, since (24) is independent of calendar time, of any forest owner factors and of any variables that may evolve in time (excluding the forest stand), successive rotation periods must have equal length. Assuming $s_i=s$ $\forall i$ leads to $pF[x(s)]-\rho[px(s)-w]/(1-e^{-\rho s})=0$. This is an equation for the Faustmann rotation, which we denote by $s=s_f$. Another argument can be based on a notion that the stationary (Faustmann) solution is unstable since $ds_{i+1}/ds_i|_{s_i=s_f}=e^{\rho s_f}>1$. Thus there cannot be any rotation programs that solve (24) and smoothly converge toward $s=s_f$. Any other possible solutions to (24) can be shown to violate second order necessary conditions requiring that the derivative of (24) with respect to s_i be nonpositive (Seierstad 1988, theorem 4, remark 8). Thus the globally optimal solution to (24) must be the Faustmann rotation.

A numerical example for the case $\delta>\rho$ is depicted in Figure 1. The optimal rotation is 70 years (Fig. 1a). Since $\delta>\rho$ the forest owner has an incentive to initially consume more than he earns, implying that the optimal consumption schedule is decreasing in time (Fig 1b). Since the capital markets are perfect, the optimal consumption path is continuous although the forest income stream and the path for financial assets are discontinuous (Fig. 1c). As $t\to\infty$ the asset approaches a cycle where the debt just equals the present value of future forest and nonforest earnings, i.e. $a(t)=-m/\rho-e^{\rho(t-t_i)}(V_f+w)$ when $t\in[t_i,t_{i+1}]$, where V_f is the Faustmann value for bare land and $e^{\rho(t-t_i)}(V_f+w)$ is the land value $t-t_i$ years after each regeneration at t_i. Given $x(0)=x_0$ in (8), the consumption time path satisfies the budget constraint $\int_0^\infty[c(t)-m]e^{-\rho t}dt-a_0-[px(s_f)-w]e^{-\rho s_f}/(1-e^{-\rho s_f})=0$ together with $U'(c)e^{-\delta t}-\mu=0$ and $\dot{\mu}=-\rho\mu$. As the development of financial assets demonstrates, this solution is inherently based on borrowing in perfect capital markets. We turn to study the rotation under binding borrowing constraints.

Figure 1. Optimal rotation and consumption without borrowing constraint



Note: $x(s)=a-be^{-cs}$, $a=600, b=842.9, c=0.012$,
$U(c)=c^{1-\alpha}/(1-\alpha)$, $\alpha=1/2$, $\delta=0.02$, $\rho=0.01$
$m=5000$, $w=3000$, $p=170$

3 Stationary rotation programs

Faustmann rotation is stationary in the sense that a constant rotation is repeated forever. We next study the properties and existence of stationary rotation programs under binding borrowing constraints.

Proposition 1. Given $\delta \leq \rho$, the stationary rotation equals the Faustmann solution and consumption is continuous. Given $\delta > \rho$ and a stationary solution, consumption jumps up at the harvesting moments but is continuous between the harvests.

Proof: Appendix 1.

Thus for forest owners with incentives to avoid exhausting their savings ($\delta \leq \rho$), the borrowing constraint cannot be binding in the stationary state and consumption is continuous, as without the borrowing constraint. However, for forest owners with incentives to consume their savings ($\delta > \rho$), the borrowing constraint becomes binding either at the date of each harvest or before the harvesting moment. The first case occurs at least with m=0, since $U'(c) \to \infty$ as $c \to 0$ rules out the possibility that c=a=0 for any interval of nonzero length. In both of these cases consumption jumps up at the harvesting moment. This shows that under a bidding borrowing constraint there is a clear connection between timber harvesting and high rates of consumption (cf. the "volvo argument").

Let us transform (23) to the current value form. Along the stationary program with borrowing constraint, successive rotations must all have the same length and same consumption schedule. Let us denote $c=c_0$ at any t_i and $\mu=\mu_0$ at t=0. By equation (11) we have $U'(c_0)=\mu_0$ at t=0 and $U'(c_0)e^{-\delta t_i}=\mu(t_i)$ at any t_i, implying $\mu(t_i)=\mu_0 e^{-\delta t_i}$ and $\mu(t_{i+1})=\mu_0 e^{-\delta(t_i+s_\infty)}$, where s_∞ is the stationary rotation. Denoting $c(t_i^-)=c_s$, $x(t_i^-)=x_s$ and multiplying (23) by $e^{\delta t_i}$ yields

$$U(c_s)+U'(c_s)(m-c_s)+U'(c_0)pF(x_s)-U(c_0)-U'(c_0)[m-c_0+\rho(px_s-w)+pe^{-\delta s\infty}F(x_s)]=0. \qquad (25)$$

When the constraint $a \geq 0$ is not binding before the end of the rotation, the budget constraint between any two successive harvests is $\int_0^{s\infty} me^{-\rho s}ds-\int_0^{s\infty} c(s)e^{-\rho s}ds+px_s-w=0$, which together with $U'(c)e^{-\delta t}-\mu=0$ and $\dot{\mu}=-\rho\mu$ defines the dependence of c_0, c_s on the rotation length $s\infty$. In the other case $\exists s'$ such that $a=0$ for $t\in[s',s\infty)$ and the budget constraint is $\int_0^{s'} me^{-\rho s}dt-\int_0^{s'} c(s)e^{-\rho s}ds+px_s-w=0$, which, together with $c(s')=m$, $U'(c)e^{-\delta t}=\mu$ and $\dot{\mu}=-\rho\mu$ for $t\in[t_i,t_i']$ defines the dependence of c_0 on $s\infty$.

Assume that in (25) $s\infty \to s°$, where $s°$ implies $px(s°)-w=0$. The LHS of (25) must approach $U'(c_0)pF[x(s°)](1-e^{-\delta s°})>0$, since, without nonforest income, $c=m$ $\forall t$ in the stationary state. When $s\infty \to \infty$ $c_s \to m$ and the LHS approaches $U(m)-U(c_0)-U'(c_0)(m-c_0)+U'(c_0)[pF(x_s)-\rho[px(\infty)-w]$, which is negative by the concavity of $U(c)$. Thus there always exists at least one level of s that satisfies (25) and where the value of the LHS of (25) is positive (negative) with levels of s lower (higher) than the optimality candidate. We have not been able to prove the uniqueness analytically, but with our functional forms there have not been any signs of multiple local optima in the numerical analysis. If there are many rotation levels satisfying the necessary conditions, the globally optimal solution is the candidate that gives the highest value for the criteria functional (Seierstand 1988, theorem 4, remark 8, or Seierstad Sydsæter 1987, theorem 13, p. 145, note 27).

Equation (25) can be interpreted in line with the Faustmann formula given by (24). The term $U'(c_0)pF(x_s)$ reflects the fact that a marginally longer rotation increases the value of the next harvest. The term $U(c_s)$ is the consumption utility from a longer period before the harvest. The term $U'(c_s)(m-c_s)$ denotes the cost of producing this prolongation by marginally redecing the rate of consumption (net of income) before the cut. Thus the sum of these terms equals the benefit of increasing the rotation length marginally. The rest of the terms show the cost of this prolongation. $U(c_0)+U'(c_0)(m-c_0)$ denotes the direct effects in consumption utility due to shorter next period. The term $U'(c_0)\rho(px_s-w)$ denotes the interest cost and $U'(c_0)pe^{-\delta s_\infty}F(x_s)$ the reduced value of the next harvest. Recall that (25) must hold between successive harvests up to infinity.

An equivalent form of the stationary cutting equation (25) is

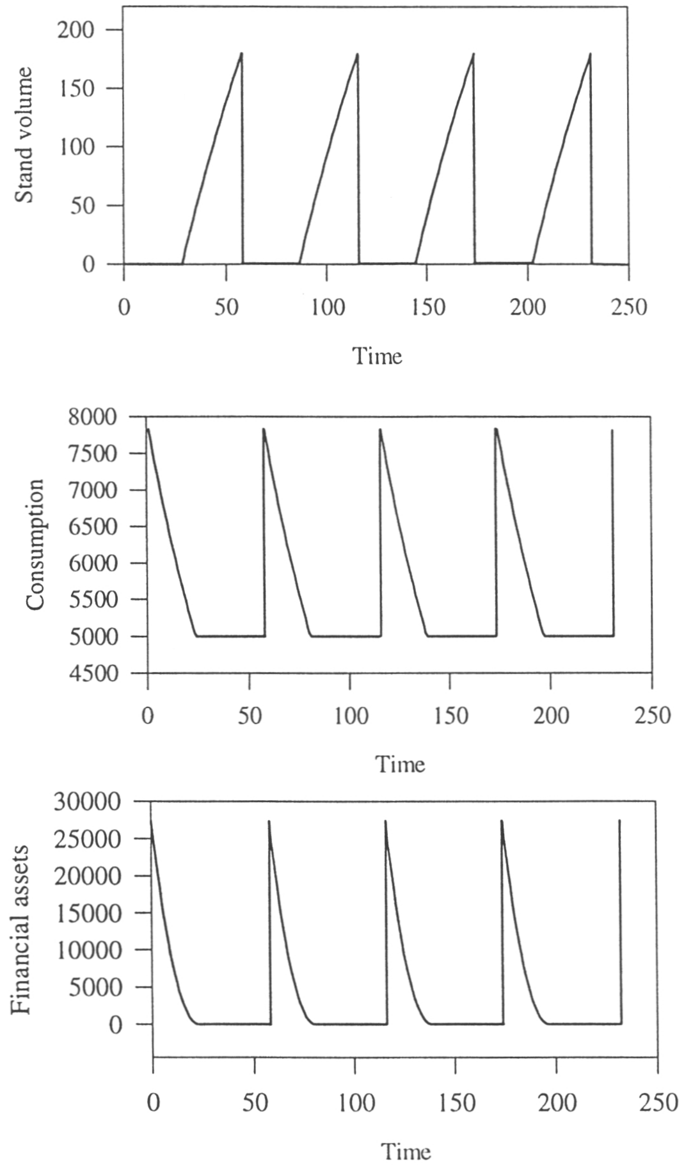
$$U'(c_0)pF(x_s)=U'(c_0)\rho(px_s-w)/(1-e^{-\delta s_\infty})-\left\{U(c_s)+U'(c_s)(m-c_s)-[U(c_0)+U'(c_0)(m-c_0)]\right\}/(1-e^{-\delta s_\infty}). \quad (26)$$

The term $U'(c_0)pF(x_s)$ is the benefit of lengthening the rotation marginally. The term $U'(c_0)\rho(px_s-w)/(1-e^{-\delta s_\infty})$ denotes the costs of postponing all the future net harvesting incomes. The remaining term, $-\{\cdot\}/(1-e^{-\delta s_\infty})$, reflects the fact that increasing the rotation length postpones all the coming jumps to a higher consumption level after a harvest. This implies another loss in present value utility. Note that the concavity of U(c) implies $-\{\cdot\}/(1-e^{-\delta s_\infty})>0$.

Equation (26) differs from the Faustmann formula $pF(x_s)=\rho(px_s-w)/(1-e^{-\rho s_f})$ in three respects. First, all monetary items are evaluated in utility units and secondly the term $U'(c_0)\rho(px_s-w)$ is divided by $(1-e^{-\delta s})$ and not by $(1-e^{-\rho s})$ as in the Faustmann formula. Third, in the Faustmann program, consumption is continuous and the last term in (26) is zero. Since with $\delta>\rho$ it holds that $(1-e^{-\delta s})>(1-e^{-\rho s})$, this difference has a positive effect on the rotation length. However, $-\{\cdot\}/(1-e^{-\delta s})>0$ implies a negative effect on rotation length. Thus we cannot directly deduce how the borrowing constraint changes the optimal rotation.

A numerical example of a stationary solution is demonstrated in Figure 2. The parameter values are the same as in Figure 1. The optimal rotation length is now 58 years (Fig. 2a). Consumption decreases between harvests and reaches the level of nonforest income 22 years after each harvest (Fig. 2b) at the same moment when financial assets reach the zero level (Fig. 2c).

Figure 2. Optimal rotation and consumption under borrowing constraint



Note: $x(s)=a-be^{-cs}$, a=600, b=842.9, c=0.012,
$U(c)=c^{1-\alpha}/1-\alpha$, $\alpha=1/2$, $\rho=0.01$, $\delta=0.02$,
m=5000, w=3000, p=170

Comparative statics of the optimal rotation period

The comparative static derivatives take different forms depending on whether the borrowing constraint becomes binding before or at the harvesting moments. In the following, we study the analytically more tractable case where financial assets reach the zero level at the harvesting moments only. The other case will be described numerically.

Proposition 2. Given $U(c)=c^{1-\alpha}/(1-\alpha)$, $0<\alpha<1$, $\delta>\rho$ *and* $a>0$ $\forall t\in[t_i,t_{i+1})$:

i) $\partial s/\partial\delta<0$ *when* $m\geq 0$ and $\rho\geq 0$ *are sufficiently small,*

ii) $\partial s/\partial m>0$,

iii) $\partial s/\partial\rho|_{m=0}>0$,

iv) $\partial s/\partial w>0$,

v) $\partial s/\partial p<0$, *given* $m>0$ or $w>0$,

vi) $\partial s/\partial p=0$, *given* $m=w=0$,

vii) $\partial s/\partial p<0$, *given* $m>0$ *and* $p/w=\sigma_1$; $\partial s/\partial p=0$, *given* $p/w=\sigma_1$ *and* $m=0$ $(\sigma_1>0)$,

viii) $\partial s/\partial p=0$, *given* $p/w=\sigma_1$, $p/m=\sigma_2$ $(\sigma_1,\sigma_2>0)$,

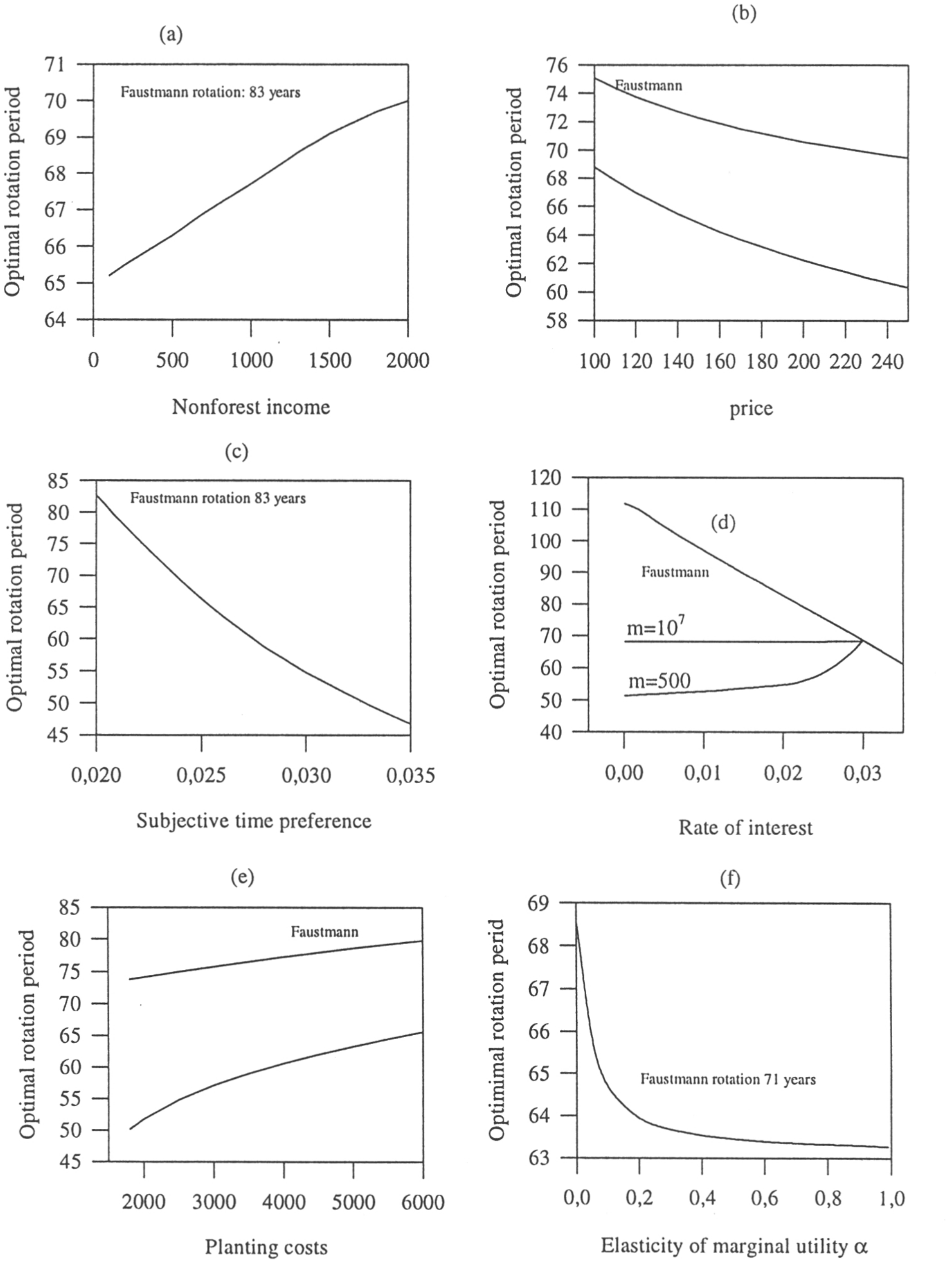
ix) $\partial s/\partial\alpha<0$.

For proof, see Appendix 2. The first result shows that increasing the rate of subjective time preference δ decreases the length of the rotation period at least when the rate of interest ρ and the level of exogenous income m are sufficiently small. Recall that δ does not enter the Faustmann formula. Since we have shown earlier that $\rho=\delta$ implies the Faustmann rotation, the result $\partial s/\partial\delta<0$ implies that the borrowing constraint shortens rotation below the Faustmann length. Although the analytical result is based on "small" ρ and m, numerical computation suggests that the result is much more general since we have found no exceptions. Next in (ii) it is shown that an increase in the nonforest income lengthens the rotation. Again recall that nonforest income is absent in the Faustmann formula. In the Faustmann model, increasing the rate of interest always shortens the rotation period. Under the borrowing constraint and zero exogenous income, it is possible to prove that the result is the reverse (iii).

In the classical Faustmann model the rotation period depends on p/w but not on the absolute levels of price or planting costs. If planting costs are zero, price does not have any effect on optimal rotation. Here the picture is slightly more complicated. An increase in the planting costs lengthens the rotation period as in the Faustmann model (iv). If either planting costs or nonforest income is nonzero, then an increase in price shortens the rotation period as in the Faustmann model (v). However, if they are both zero, price does not affect optimal rotation (vi). If timber price and planting costs increase, compared to the (nonzero) nonforest income, the rotation period decreases (vii). If timber price, planting costs and nonforest income change at the same rate, the rotation period remains constant (viii). Finally, increasing the elasticity of marginal utility α shortens the rotation period. This is natural since with higher marginal utility, the forest owner prefers a more even consumption profile, which is obtained via shorter rotations and more frequent harvesting.

Figure 3 shows numerical results on the dependence of the rotation period on parameter values. The computation covers examples where financial assets reach the zero level at the harvesting moment or before harvesting. Fig. 3a shows that the rotation period depends positively on nonforest income. In addition, it shows an example where the rotation period with borrowing constraint is shorter than the Faustmann length (83 years). Increasing nonforest income to $m=10^7$ lengthens the rotation period to 75.7, still shorter than the Faustmann rotation. Fig. 3b shows the rotation period as a function of price and Fig. 3c as a function of the subjective time preference. Proposition 2iii shows analytically that the rotation period is an increasing function of the rate of interest, given m=0. Fig. 3d shows that with the parameter values used, an increase in the rate of interest implies a longer rotation period for a wide range of nonforest income. For example, when $m=10^7$ and $\rho=10^{-5}$, the rotation period is 68.039 compared to the Faustmann rotation, 113.204 years. Increasing the rate of interest to $\rho=0.0299$ lengthens the rotation period to 68.585. Recall

Figure 3. Comparative statics of optimal rotation under borrowing constraint



The parameter values are as follows:
$x(s)=K/(1-Ce^{-rs})$, $C=(x_0-K)/x_0$, $K=500$, $r=0.048$, $x_0=10$,
$U(c)=c^{1-\alpha}/(1-\alpha)$, $0<\alpha<1$.

a: $\rho=0.02$, $\delta=0.025$, $p=170$, $w=3000$, $\alpha=1/2$
b: $\rho=0.028$, $\delta=0.03$, $w=3000$, $m=500$, $\alpha=1/2$
c: $\rho=0.02$, $p=170$, $w=3000$, $m=500$, $\alpha=1/2$
d: $\delta=0.03$, $p=170$, $w=3000$, $m=500$, $\alpha=1/2$
e: $\rho=0.025$, $\delta=0.03$, $p=170$, $m=0$, $\alpha=1/2$
f: $\rho=0.028$, $\delta=0.03$, $p=170$, $w=3000$, $m=500$

that with $\rho \geq \delta = 0.03$ the stationary rotation period with the borrowing constraint equals the Faustmann rotation (Proposition 1). This implies that the rotation period may be a nonmonotonic function of the rate of interest. Fig. 3e shows that higher planting costs lengthen the rotation period as in the Faustmann case. Finally Fig. 3f shows how the rotation period depends on the elasticity of marginal utility, α.

The value of forest land and incentives for stand regeneration

Given a stationary forest rotation program, the forest owners maximized utility W_1 is

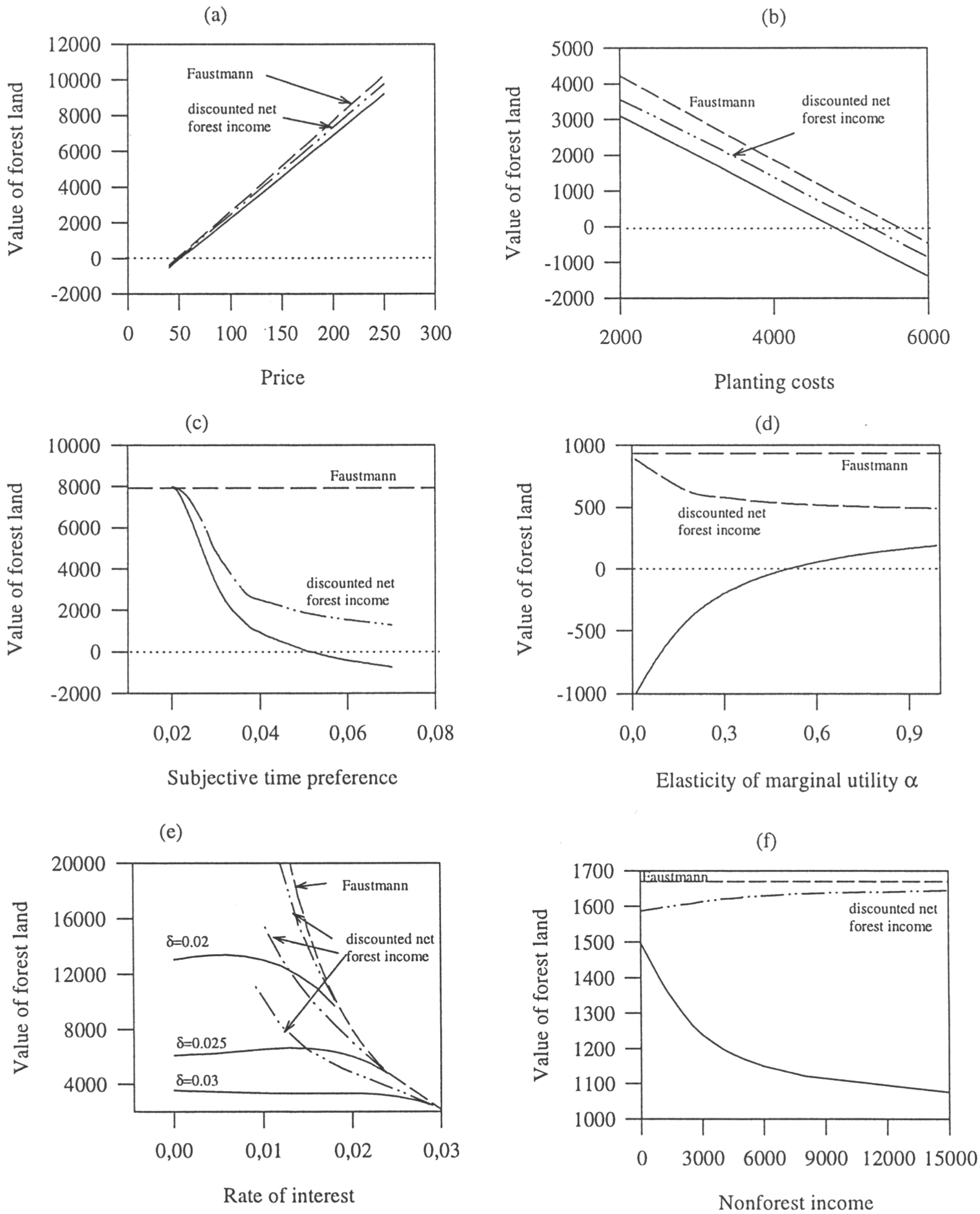
$$W_1 = \int_0^{s\infty} U(c*)e^{-\delta s} ds / (1 - e^{-\delta s\infty}),$$

where $s\infty$ and $c*$ are the optimal stationary rotation period and consumption schedule respectively. If the owner sells his forest land, the maximized utility is defined as

$$W_2 = \max_{\{c\}} \int_0^{\infty} U(c)e^{-\delta t} dt, \text{ s.t. } \dot{a} = \rho a - c + m, \; a(0) = px(s\infty) + V, \text{ and } a(t) \geq 0,$$

where V is the value of bare forest land given that $W_1 = W_2$. Thus the value of forest land, i.e. forest owner's reservation price, equals the monetary compensation that makes the maximized welfare without the forest land equal to the maximized welfare based on both forest and nonforest income. Without the binding borrowing constraint, the value of forest land equals the present value net forest income (ρ as the rate of discount), i.e. the Faustmann land value. With the borrowing constraint, the rotation period deviates from the Faustmann rotation and thus the present value net forest income must also deviate from the Faustmann land value. However, under the borrowing constraint, discounting based on market rate of interest does not yield the true value of some future payment to the decision maker (see Hirshleifer 1970, p. 196). Thus we can expect that under the borrowing

Figure 4. The value of forest land under borrowing constraint



Note: The Functions and parameter values are as follows: $x(s)=K/(1-Ce^{-rs})$, $C=(x_0-K)/x_0$, K=500, r=0.048, $x_0=10$, $U(c)=c^{1-\alpha}/(1-\alpha)$,

a: δ=0.025, ρ=0.022, w=2000, m=0, α=1/2
b: δ=0.03, ρ=0.025, p=170, m=0, α=1/2
c: ρ=0.02, p=170, w=2000, m=0, α=1/2
d: δ=0.03, ρ=0.025, p=170, w=4770, m=0
e: δ=0.025, p=170, w=2000, m=0, α=1/2
f: δ=0.03, ρ=0.028, p=170, w=3000, α=1/2

constraint $V \neq [px(s_\infty)e^{-\rho s_\infty}-w]/(1-e^{-\rho s_\infty})$. It is possible to study the forest land value analytically using the envelope theorem and differentiating W_1-W_2=0. However, we present the numerical examples shown in Figure 4.

The Faustmann land value depends positively on timber price p and negatively on planting costs w and the rate of interest ρ. Figs. 4a and 4b show examples where similar dependencies hold for p and w under the borrowing constraint. The examples also show how the borrowing constraint yields a deviation between the true land value and the discounted net forest income. It is a priori clear that the borrowing constraint must reduce the land value. As shown, this yields the result that low timber price or high planting costs imply negative land values at parameter levels where the Faustmann land value is positive (e.g. Figs. 4a and 4b). Thus the borrowing constraint reduces the economic incentives to replant a new stand after the harvest.

Increasing the rate of subjective time preference reduces the value of forest land (Fig. 4c). A forest owner with high elasticity of marginal utility (high α) prefers an even consumption time path. According to Fig. 4d also his reservation price for the forest land is higher than for forest owners with low α and more uneven consumption between rotations. Note that in Fig. 4d the Faustmann land value is about 1000. With low levels of α, the discounted net forest income under the borrowing constraint is 950. However, the true value of forest land under the borrowing constraint is -1000. Thus the forest owner has no economic incentives for replanting, although the classical rotation model clearly suggests the reverse.

The Faustmann land value decreases with the rate of interest, and as $\rho \to 0$ the land value approaches infinity. Examples in Fig. 4e suggest that under a borrowing constraint this dependence may be nonmonotonic and the value of forest land remains bounded when the rate of interest approaches zero. Again the difference between discounted net forest income and true land value may be large. Finally, Fig. 4f shows an example where the land value decreases as a function of nonforest income. We have not found numerical counterexamples for this somewhat surprising result. This means that, with higher nonforest income, unrestricted borrowing is more important than with lower nonforest income in aiming at maximum utility from an even aged forest stand. In other words, the borrowing constraint affects land value more in the case of a forest owner with higher than lower income. It is easy to find examples where land value decreases below zero due to high nonforest income. Thus, under the borrowing constraint, high nonforest income may yield low incentives for regeneration.

4 Nonstationary rotation programs

The rotation difference equation and its linear approximation

Equation (23) can be written in current value form as

$$\Gamma=U[c(t_i^-)]+U'[c(t_i^-)][m-c(t_i^-)]+pU'[c(t_i)]F[x(s_i)]-$$
$$U[c(t_i)]-U'[c(t_i)]\{\rho[px(s_i)-w]+m-c(t_i)\}-pU'[c(t_{i+1})]e^{-\delta s_{i+1}}F[x(s_{i+1})]=0. \quad (27)$$

As noted in section 2, this is a nonlinear difference equation. To find its order, we must study how $c(t_i^-)$, $c(t_i)$ and $c(t_{i+1})$ depend on the length of different successive rotations. Consumption level $c(t_i^-)$ depends on s_i but also on s_{i-1} since the length of s_{i-1} determines the initial assets at the beginning of s_i. Similarily, $c(t_i)$ depends on the initial asset level at t_i and thus on s_i but also on the length of the coming rotation s_{i+1}. Analogously $c(t_{i+1})$ depends on s_{i+1} and s_{i+2}. Thus (27) is a third order difference equation which, if computed forwards, determines s_{i+2} if s_{i-1}, s_i and s_{i+1} are known. In the previous section, we studied stationary rotation programs as steady state solutions to (27). Next we study the stability properties of the stationary program and then compute nonstationary rotation programs.

We restrict the local stability analysis to cases where assets reach zero level just at the harvesting moment (m is "small"). The budget constraint between any two rotations is

$px(s_{i-1})$-w+ $\int_0^{s_i} me^{-\rho t}dt - \int_0^{s_i} c(t)e^{-\rho t}dt=0$. Assuming $U(c)=c^{1-\alpha}/(1-\alpha)$, $0<\alpha<1$, the budget constraint implies

$$c(t_i)=e^{\delta s_{i+1}/\alpha}(\alpha\rho+\delta-\rho)\{e^{\rho s_{i+1}}[p\rho x(s_i)+m-\rho w]-m\}/\{\alpha\rho[e^{\delta s_{i+1}/\alpha+\rho s_{i+1}}-e^{\rho s_{i+1}/\alpha}]\}, \quad (29)$$

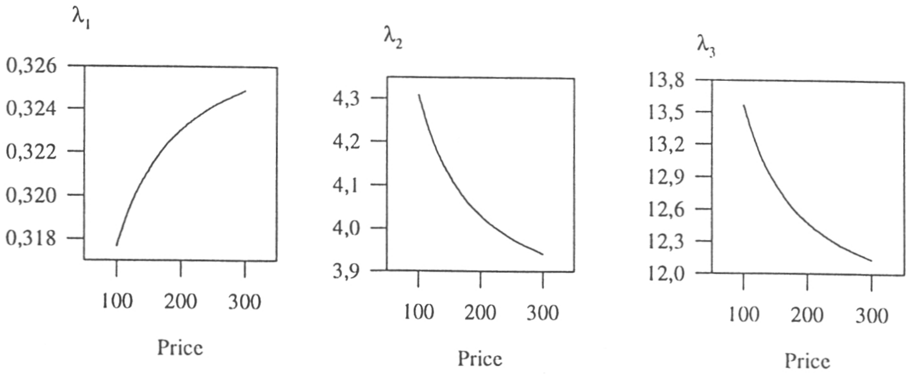
$$c(t_i^-)=c(t_{i-1})e^{s_i(\rho-\delta)/\alpha}. \quad (30)$$

Equation (29) readily suggests how to construct $c(t_{i+1})$. We can now eliminate $c(t_i^-)$, $c(t_i)$ and $c(t_{i+1})$ from (27), which is then a function of s_{i-1}, s_i, s_{i+1} and s_{i+2}. To study the local stability properties of the stationary state, we compute the derivatives: $\partial\Gamma/\partial s_{i-1}\equiv\zeta_0$, $\partial\Gamma/\partial s_i\equiv\zeta_1$, $\partial\Gamma/\partial s_{i+1}\equiv\zeta_2$, $\partial\Gamma/\partial s_{i+2}\equiv\zeta_3$. The characteristic equation is $\zeta_3\lambda^3+\zeta_2\lambda^2+\zeta_1\lambda+\zeta_0=0$. Consider the possibility that all the characteristic roots are real and $0<\lambda_1<1$, $\lambda_2>1$, $\lambda_3>1$. In such a case, the stationary state has a local saddle point property.

The saddle point property can be interpreted as follows: assume some given initial level a_0 for the financial assets. When we now specify equation (27) for the first harvest at t_1, the equation includes three unknowns: s_1, s_2, and s_3. After specifying some s_1 and s_2, (27) determines s_3. The (local) saddle point property means that there can exist only one pair (s_1,s_2) such that solving for s_3 and then s_i for $i=4,...,\infty$ yields a convergence toward the stationary program. Numerical computation of the characteristic roots suggests that equation (27) possesses this property. Figure 5 shows examples of characteristic roots as a function of timber price where $0<\lambda_1<1$, λ_2, $\lambda_3>1$ in all cases.

Close to the stationary state, we can write along the saddle point solution that $\Delta s_i \simeq q\lambda_1{}^i$, where q is a constant. Thus if $\Delta s_i=\underline{\Delta}$ is a deviation from the stationary rotation s_∞, it holds that $\Delta s_{i+1}=\underline{\Delta}\lambda_1$ and $\Delta s_{i+2}=\underline{\Delta}\lambda_1{}^2$. It is now possible to compute Δs_{i-1} from (27) and then proceed backwards. The approximation will be arbitrarily close to the original solution for the nonlinear equation when $\underline{\Delta}$ is sufficiently small.

Figure 5. The characteristic roots of the linearized harvesting equation



Note: $x(s)=600-842.9e^{-0.012s}$, $U(c)=c^{1-\alpha}/1-\alpha$, $\alpha=0.9$
$\delta=0.03$, $\rho=0.015$, $p=170, w=3000, m=0$.

Examples of nonstationary rotations

We hypothesize that in our model it is possible to distinguish the following four cases:

1: If $a(0)\geq px(s_f)-w$ and $\delta\leq\rho$, then there is no effective borrowing constraint and the Faustmann rotation is optimal $\forall s_i$, $i=1,\ldots,\infty$.

2: If $a(0)<px(s_f)-w$ and $\delta\leq\rho$, Faustmann rotation is reached from below in finite time if $\delta<\rho$ and in infinite time if $\delta=\rho$.

3: If $a(0)>px(s_\infty)-w$ and $\delta>\rho$, the rotation period decreases toward the stationary rotation, $s_\infty<s_f$.

4: If $a(0)<px(s_\infty)-w$ and $\delta>\rho$, the rotation period increases toward the stationary rotation, $s_\infty<s_f$.

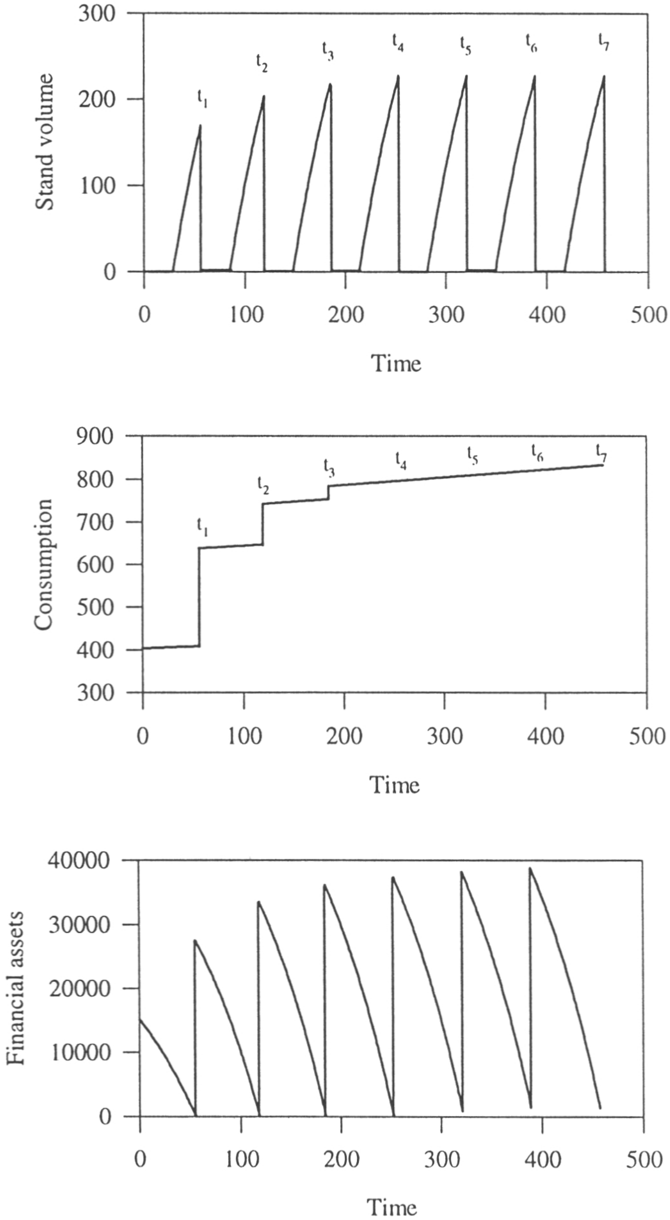
We have already shown that in the case where $\delta<\rho$ the stationary rotation program must be the Faustmann rotation (Proposition 1). If the forest owner's initial assets (at the moment when he has bare land) are not lower than $px(s_f)-w$, the credit rationing constraint will never be binding. If $a_0<px(s_f)-w$ (case 2), the optimal solution is initially constrained by the borrowing constraint and thus the solution initially deviates from the Faustmann rotation.

However, since $\delta<\rho$ implies savings incentives, the optimal solution must approach the Faustmann rotation when the assets accumulate in time. When $\delta<\rho$ the optimal consumption level increases between harvests. If the rotation period approaches the Faustmann rotation period asymptotically, it should hold that $a(t_i^-)=0$ $\forall i$, since otherwise μ would be continuous (by 16) at the harvesting moment and the rotation could not deviate from the Faustmann lenght. However, $a(t_i^-)=0$ $\forall i$, monotonically increasing consumption between the harvests, upward jumps in consumption at harvesting moments, together with a rotation program that asymptotically converges toward the Faustmann solution, is a contradiction. Thus with $\delta<\rho$ the optimal rotation must reach the Faustmann length in finite time. Accordingly, it can be shown that with $\delta=\rho$ there must be asymptotic convergence toward the Faustmann rotation.

Figure 6a-c shows a numerical example for case 2 and $\delta<\rho$. It has been computed backwards by first solving the optimal rotation and consumption program, given $a(t_4)\geq px(s_f)-w$. With this initial asset level the optimal solution must continue as a Faustmann program $\forall t\in[t_4,\infty)$. Since s_4, s_5 and s_6 equal the Faustmann length, it is possible to proceed backwards using (27). This yields a rotation program as in Fig. 6a where $s_1\simeq48$, $s_2\simeq55$, $s_3\simeq62$, $s_4=s_5=s_6\simeq67$. Consumption increases discontinuously until the first harvest with Faustmann rotation occurs. Financial assets increase discontinuously but remain strictly positive after the switch to the Faustmann rotation.

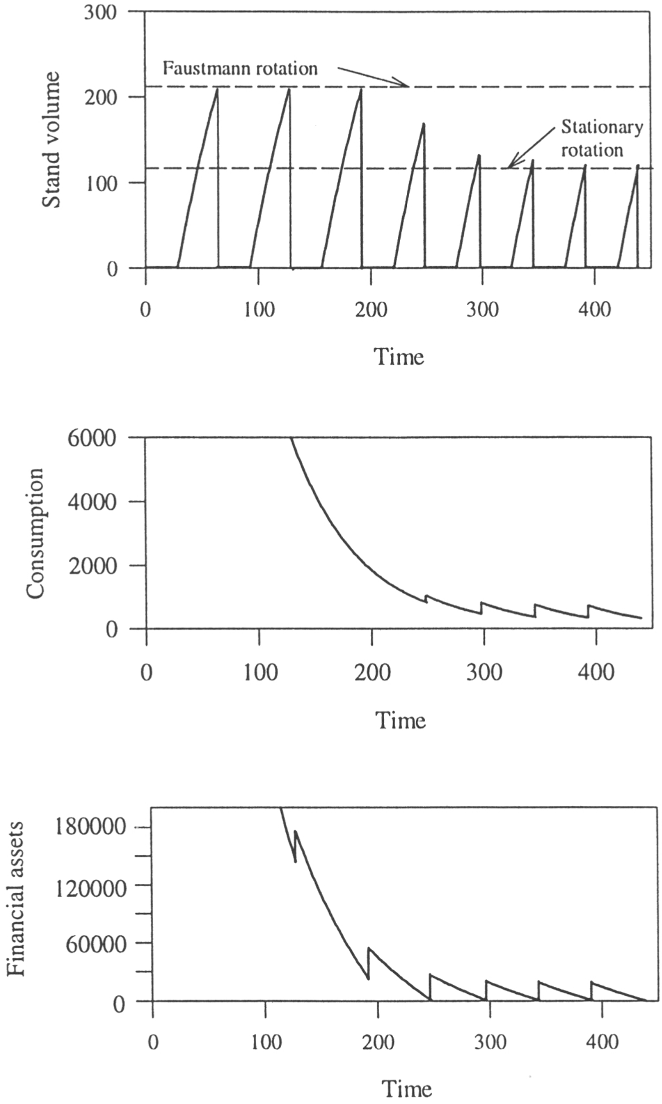
Figure 7a-c shows a numerical example of case 3. The given parameter values imply that the Faustmann rotation is $s_f\simeq64.338$ while the stationary rotation under the borrowing constraint is $s_\infty\simeq46.849$. The characteristic roots for the linearized harvesting equation are: $\lambda_1=0.322066$, $\lambda_2=4.07741$, $\lambda_3=12.6601$. As a deviation from the stationary program in backward computation, we use $\underline{\Delta}=10^{-7}$. After 14 rotations backwards the rotation length equals about 56 years. Proceeding backwards using equation (27) yields a solution where consumption would jump downwards at the harvesting moment. This violates condition (16), suggesting that there must be a switch to a regime where consumption remains

Figure 6. Optimal rotation and consumption with low initial assets and δ<ρ.



Note: $x(s)=a-be^{-cs}$, a=600, b=842.9, c=0.012
$U(c)=c^{1-\alpha}/(1-\alpha)$, $\alpha=0.9$, $\delta=0.01$, $\rho=0.0102$,
m=50, w=1000, p=170.

Figure 7. Optimal rotation and consumption with high initial assets and with $\delta > \rho$.



Note: $x(s)=600-842.9e^{-0.012s}$, $U(c)=c^{1-\alpha}/(1-\alpha)$, $\alpha=0.9$, $\delta=0.03$, $\rho=0.015$, $p=170$, $w=3000$, $m=0$.

continuous and financial assets strictly positive at the harvesting moments. Taking this into account, the length of the next period is 64.193 years. Continuing backwards then yields converge toward the Faustmann rotation. Thus, when looking forward, the forest owner has high initial assets and, during the early rotations, financial assets decrease but remain positive and consumption is continuous. However, the rotation period is shorter than the Faustmann rotation and decreases in time. After a finite number of rotations, the borrowing constraint becomes binding, consumption jumps up at the harvesting moments and the rotation period converges toward the stationary program as $t \to \infty$. Applying a similar procedure but assuming $\underline{\Delta}=-10^7$ as the initial deviation, it is possible to compute examples for case 4 where the initially short rotation increases asymptotically toward $s_\infty \simeq 46.849$.

5 Discussion

Empirical testing of our model is beyond the scope of the present paper, but a few observations on the evidence from earlier studies are in order. The overwhelming evidence suggesting that forest harvesting decisions depend on forest owner characteristics has perhaps cast the darkest shadow over the Faustmann model. According to a number of studies, a parametric increase in nonforestry income tends to increase optimal rotation period, as indicated by less frequent harvests (Binkley 1981, Romm et al. 1989, Dennis 1989, 1990, Hyberg and Holthausen 1989). This is clearly in line with our comparative static results (Proposition 2ii). On the other hand, the effect of permanent income on harvest per land unit should be positive. This has been found to be the case in Kuuluvainen and Salo (1991), Kuuluvainen et al. (1996) and Kuuluvainen and Tahvonen (1996).

Under a borrowing constraint, an immediate effect of a parametric shift in the interest rate is negative but the steady state effect is positive (Proposition 2iii). A negative effect for the interest rate has been estimated in some Scandinavian studies using aggregate data (Toppinen and Kuuluvainen 1997) or micro data with information on interest rates on individual forest owners (Pajuoja 1994) or using market interest rates (Kuuluvainen and Tahvonen 1997). Another basic observation in empirical studies is the dependence of timber supply on the forest owner's age. A typical observation is that frequency of sales decrease over the owner's age (e.g. Romm et al. 1989). Our model explains this observation under a binding borrowing constraint and accumulating financial assets.

It may be typical for household forest owners that lending and borrowing rates of interest differ. Taking this into account, instead of a borrowing constraint, will add one more state variable to the model but may still be analytically tractable. Our sensitivity analysis for the rotation period and land value is restricted to stationary states and should be generalized. It should be possible to compute the dependence of the rotation length given any initial level for financial assets and stand age. Allowing intertemporal variation in prices and nonforest income would be highly relevant for empirical purposes. Finally, we note that existing studies adding stochastic features to the Faustmann model are restrictive since they assume profit maximization and then implicitly relay on the Fisherian separation theorem.

Appendix 1. Proof of proposition 1.

Assume $\delta \leq \rho$. In (16), $\partial W_{i+1}/\partial a(t_i^-)=\mu(t_i)$. If μ is discontinuous at t_i, (16) requires that μ must jump downwards and $a(t_i^-)=0$ $\forall i$. Assume that $a(t)>0$ $\forall t\in[t_i,t_{i+1})$, implying that we can set $\tau=0$. By (11) and (14) together with $\delta \leq \rho$ it follows that $\dot{c}\geq 0$ for all $t\in[t_i,t_{i+1})$. If μ jumps down at t_i, it follows that c jumps up. A constant rotation period, $\dot{c}\geq 0$ $\forall t\in[t_i,t_{i+1})$ and a jump up in c at t_i $\forall i$ is clearly inconsistent with $a(t_i^-)=0$ $\forall i$. Assume next that $\exists t_i'$ such that $t_i'\in(t_i,t_{i+1})$ and that $a(t)=0$ for $t\in[t_i',t_{i+1})$. When $t\in[t_i,t_i')$, we have $a(t)>0$, $\tau=0$ and $\dot{c}>0$. At t_i' there must be a jump down to c=m. Since c is discontinuous at t_i' also $\dot{a}$ is discontinuous implying that τ must be continuous (12ii). Since μ^* must be continuous, condition (13i) implies that also μ must be continuous at t_i'. Continuous μ with discontinuous c contradicts (11). Thus, when $\delta \leq \rho$, a stationary program is inconsistent with discontinuous μ. Continuous μ implies that (23) reduces to (24), implying $s_i=s_f$ for $\forall i$.

Assume next that $\delta>\rho$. If $a>0$ for $t\in[t_i,t_{i+1})$, then μ and c are clearly continuous between harvests. With $\delta>\rho$ and continuous c over the cutting moment, we obtain $c\rightarrow 0$ as $t\rightarrow\infty$ together with $a\rightarrow\infty$ as $t\rightarrow\infty$ due to forest and nonforest income. This cannot constitute the globally optimal solution. Thus μ must be discontinuous at t_i and by (16) it jumps down, implying that c jumps up at each t_i. Assume next that there exists t_i' after which c=m and a=0 within each rotation. For $t\in[t_i,t_i')$, it holds that $a>0$, τ constant, e.g. $\tau=0$, $\mu=\mu^*-\tau$, $\mu=U'(c)e^{-\delta t}$ and $c>m$. For $t\in[t_i',t_{i+1})$, we have $\mu=U'(m)e^{-\delta t}$ and c=m. By (13i) it holds that $\tau=\mu^*-U'(m)e^{-\delta t}$ and by (14) that $\mu^*(t)=\rho U'(m)e^{-\delta t}/\delta+U'(m)e^{-\delta t_i'}(1-\rho/\delta)$ when $t\in[t_i',t_{i+1})$. This yields by (13i), that $\tau=(\rho/\delta-1)U'(m)e^{-\delta t}+U'(m)e^{-\delta t_i'}(1-\rho/\delta)$, implying $\dot{\tau}=-\delta U'(m)e^{-\delta t}(\rho/\delta-1)>0$. Thus τ is nondecreasing, as required. Since (12i, 12ii, 13i, 13ii and 14) are satisfied by construction, the solution satisfies the necessary conditions. If c were discontinuous at t_i', then by (12ii) τ is continuous and therefore by (13i), (13ii) and (14) also μ is continuous at t_i', contradicting the discontinuity of c.■

Appendix 2. Proof of proposition 2.

In proving the comparative static derivatives we apply a modification of a well known lemma in optimal control theory (see Pontryagin et. al. 1962, p. 122-123). It states that the function $f(s)=\sum_{i=1}^{n} b_i(s)e^{a_i s}$, where $b_1(s),...,b_n(s)$ are polynomials of degree $r_1,...,r_n$, respectively, there can be at most $r_1+r_2+...+r_n+n-1$ zeros. For our purposes we need to extend the lemma as follows:

Definition: s_0 is a zero of the function f with multiplicity k if $f(s_0)=f'(s_0)=,...,f^{(k-1)}(s_0)=0$ but $f^{(k)}(s_0)\neq 0$.

Lemma 1: Let $f_k(s)=\sum_{i=1}^{k} b_i(s)e^{a_i s}$, $f_k(s)$ not identically zero, where $b_i(s)$ is a polynomial of degree r_i, i=1,..., k and $a_1,...,a_k$ are constants. Then the sum of multiplicities of the zeros of f_k is at most $\sum_{i=1}^{k} r_i+k-1$.

Proof: Clearly the functions g(s) and $h(s)=e^{a_1 s}g(s)$, where a_1 constant, have the same zeros. These zeros have also the same multiplicities. To show this, let τ be a zero of g(s) with multiplicity m, i.e. $g(\tau)=g'(\tau)=...=g^{(m-1)}(\tau)=0$, but $g^{(m)}(\tau)\neq 0$. Then $h(\tau)=0$,

$$h'(s)=e^{a_1 s}[a_1 g(s)+g'(s)],$$
$$h''(s)=e^{a_1 s}[a_1^2 g(s)+2a_1 g'(s)+g''(s)],$$
$$\vdots$$
$$h^{(j)}(s)=e^{a_1 s}[\sum_{i=0}^{j}\binom{j}{i}a_1^{j-i}g^{(i)}(s)],$$

and therefore $h'(\tau)=...=h^{(m-1)}(\tau)=0$, but $h^{(m)}(\tau)=e^{a_1\tau}g^{(m)}(\tau)\neq 0$, which proves the claim.

The observation made above immediately proves the lemma for k=1. Next, note that if τ is a zero of g(s) with multiplicity m, then τ is a zero of g'(s) with multiplicity m-1 and it holds that between any two consecutive zeros of g(s) there must be a zero of g'(s). Thus the sum of multiplicities of the zeros (SMZ) of g'(s) is at least m-1, where m is the SMZ of g(s). By repetition, the SMZ of $g^{(j)}$ is at least m-j. Let the SMZ of functions of type f_k be at most m_k and $g(s)=f_k(s)+b(s)$, where b(s) is a polynomial of degree r. Let the SMZ of g be m. Then $g^{(r+1)}(s)=d^{r+1}f_k/ds^{r+1}=\tilde{f}_k$, a new function of type f_k. Then the SMZ of $g^{(r+1)}\geq m-r-1$ and

$g^{(r+1)} \leq m_k$. Therefore $m \leq m_k + r + 1$. But the same is true for f_{k+1} because

$$f_{k+1}(s) = f_k(s) + b_{k+1}(s)e^{a_{k+1}s} = e^{a_{k+1}s}[\tilde{f}_k(s) + b_{k+1}(s)]$$

and thus f_{k+1} and $\tilde{f}_k + b_{k+1}$ have the same zeros with the same multiplicities. The lemma is true for k=1 and by repetition, also true for all $k \geq 1$.■

We can now proceed by studying the signs of the comparative static derivatives. We first give equation (25) a more compact form. Given $U(c) = c^{1-\alpha}/(1-\alpha)$, where $0<\alpha<1$, the terms $U(c_s) + U'(c_s)(m-c_s) - U(c_0) - \lambda_0(m-c_0)$ in equation (25i) can be written as

$$c_s^{1-\alpha}/(1-\alpha) + c_s^{-\alpha}(m-c_s) - c_0^{1-\alpha}/(1-\alpha) - c_0^{-\alpha}(m-c_0) =$$
$$-c_0^{-\alpha}\{\alpha c_0 e^{gs}/(\alpha-1) - me^{\delta s - \rho s} - [\alpha(c_0-m)+m]/(\alpha-1)\},$$

where $g=(1-1/\alpha)(\delta-\rho)<0$ (by $\delta>\rho$ and $0<\alpha<1$) and $c_s = c_0 e^{s(\rho-\delta)/\alpha}$. Write equation (25i) in the form $(c_0)^{-\alpha} p\Gamma_1 = 0$, where

$$\Gamma_1 \equiv (c_0/p)\alpha(1-e^{gs})/(\alpha-1) - m(1-e^{\delta s - \rho s})/p + F[x(s)](1-e^{-\delta s}) - \rho[x-w/p] = 0, \text{ (A1)}$$

and where $c_0 = e^{\delta s/\alpha}(\alpha\rho+\delta-\rho)[e^{\rho s}(a_0\rho+m)-m]/[\alpha\rho(e^{\delta s/\alpha+\rho s} - e^{\rho s/\alpha})]$ and $a_0 = px(s) - w$. The comparative static derivatives can now be computed from $\Gamma_1 = 0$ in (A1).

i) We first study $\partial s/\partial\delta = -(\partial\Gamma_1/\partial\delta)/(\partial\Gamma_1/\partial s)$. Note that $\Gamma_1 = 0$ is a first order condition for a relative maximum. The second order necessary condition is $\partial\Gamma_1/\partial s \leq 0$. We can exclude the case $\partial\Gamma_1/\partial s = 0$ if we find that $\partial\Gamma_1/\partial\delta \neq 0$ since $\partial\Gamma_1/\partial s = 0$, together with $\partial\Gamma_1/\partial\delta \neq 0$, will not be an equilibrium with any normal smooth comparative static properties. Thus the $\text{sgn}(\partial s/\partial\delta) = \text{sgn}(\partial\Gamma_1/\partial\delta)$. Differentiating Γ_1 yields

$$\partial\Gamma_1/\partial\delta = c_0 b\alpha(1-e^{gs})/[p(\alpha-1)] - c_0\alpha(-s/\alpha+s)e^{gs}/[p(\alpha-1)] + sme^{\delta s-\rho s}/p + sF[x(s)]e^{-\delta s}, \quad \text{(A2)}$$

where b is defined by $\partial c_0/\partial\delta \equiv c_0 b = c_0\{s/\alpha+1/(\alpha\rho+\delta-\rho)-se^{\delta s/\alpha+\rho s}/[\alpha(e^{\delta s/\alpha+\rho s}-e^{\rho s/\alpha})]\}$. Eliminate F[x(s)] by solving it from from (A1). Some rearrangement yields $\partial\Gamma_1/\partial\delta=$

$$c_0\Big\{b\alpha(1-e^{gs})/(\alpha-1)-\alpha(-s/\alpha+s)e^{gs}/(\alpha-1)+se^{-\delta s}\alpha e^{gs}/[(\alpha-1)(1-e^{-\delta s})]-$$
$$s\alpha e^{-\delta s}/[(\alpha-1)(1-e^{-\delta s})]\Big\}/p+ \qquad (\equiv c_0\Gamma_2/p)$$
$$sme^{\delta s-\rho s}/p-sme^{-\rho s}/[p(1-e^{-\delta s})]+se^{-\delta s}m/[p(1-e^{-\delta s})]+\rho[x(s)-w/p]se^{-\delta s}/(1-e^{-\delta s}). \qquad \text{(A3)}$$

The last four terms in (A3) approach zero as $\rho\to 0$ and $m\to 0$. Since $c_0>0$, we can prove claim i) by showing that $\Gamma_2<0$ in (A3). We obtain

$$\Gamma_2=\alpha(1-e^{gs})/(\alpha-1)\{b+s(1/\alpha-1)e^{gs}/(1-e^{gs})-se^{-\delta s}/(1-e^{-\delta s})\}\equiv\alpha(1-e^{gs})/(\alpha-1)\Gamma_3. \qquad \text{(A4)}$$

Since $\alpha(1-e^{gs})/(\alpha-1)<0$, the task is to show that $\Gamma_3>0$. Using the definition of b it follows that

$$\Gamma_3\alpha/s=1+\alpha/[s(\alpha\rho+\delta-\rho)]+e^{\delta s/\alpha+\rho s}/(e^{\rho s/\alpha}-e^{\delta s/\alpha+\rho s})+(1-\alpha)e^{gs}/(1-e^{gs})+\alpha/(1-e^{\delta s}). \qquad \text{(A5)}$$

We obtain $\alpha/[s(\rho\alpha-\rho+\delta)]+\alpha/(1-e^{\delta s})=\alpha[e^{\delta s}-\delta s-1+\rho s(1-\alpha)]/[(e^{\delta s}-1)(\alpha\rho s-\rho s+\delta s)]$. The denominator is clearly positive. The numerator is positive since it is increasing with δ and positive with $\delta=0$. The remaining terms of (A5), i.e $1+e^{\delta s/\alpha+\rho s}/(e^{\rho s/\alpha}-e^{\delta s/\alpha+\rho s})+(1-\alpha)e^{gs}/(1-e^{gs})$, can be written as

$$\Gamma_4\equiv e^{\rho s/\alpha}\{e^{\delta s/\alpha+\rho s}[e^{\delta s}(\alpha-1)+1]-\alpha e^{\rho s/\alpha+\delta s}\}/[(e^{\delta s/\alpha+\rho s}-e^{\rho s/\alpha})(e^{\rho s/\alpha+\delta s}-e^{\delta s/\alpha+\rho s})].$$

The denominator of Γ_4 is clearly negative. The nominator can be written as $u_1=e^{\rho s/\alpha+\delta s/\alpha+\rho s}f_3(s)$, where $f_3(s)=-\alpha e^{s(\rho-\delta)/\alpha+\delta s-\rho s}+e^{\delta s}(\alpha-1)+1$. This yields $f_3(0)=0$, $f_3'(0)=\rho(\alpha-1)<0$, and $\lim_{s\to\infty} f_3(s)=-\infty$. Thus s=0 is a zero with multiplicity of 1. Since $f_3(s)$ is negative for small and large levels of s, the possibility that $f_3(s)>0$ for some level of s would contradict lemma 1. Thus $f_3(s)<0$ and $u_l<0$ for $\forall s>0$, implying that $\Gamma_4>0$, $\Gamma_3>0$, $\Gamma_2<0$, $\partial\Gamma_1/\partial\delta<0$ and $\partial s/\partial\delta<0$ when ρ and m are sufficiently small.

(ii) We next study $\partial s/\partial m$. Computing and some rearrangement yields from (A1) that

$$\partial\Gamma_1/\partial m=e^{-\rho s}\Big\{e^{\delta s/\alpha+\rho s}[\rho e^{\delta s}(\alpha-1)+\delta e^{\rho s}-\alpha\rho-\delta+\rho]-e^{\rho s/\alpha}\{e^{\delta s}[e^{\rho s}(\alpha\rho+\delta-\rho)-\delta]+\rho e^{\rho s}(1-\alpha)\}\Big\}/[p\rho(\alpha-1)(e^{\delta s/\alpha+\rho s}-e^{\rho s/\alpha})]. \qquad \text{(A6)}$$

The denominator is negative. The numerator equals $f_6(s)/e^{\delta s/\alpha}$, where

$$f_6(s)=\delta e^{s(\rho-\delta)/\alpha+\delta s-\rho s}-(\alpha\rho+\delta-\rho)e^{s(\rho-\delta)/\alpha+\delta s}+(\alpha\rho-\rho)e^{s(\rho-\delta)/\alpha}+(\alpha\rho-\rho)e^{\delta s}+\delta e^{\rho s}-\alpha\rho-\delta+\rho.$$

Because $e^{\delta s}$ is the fastest growing term, $\lim_{s\to\infty} f_6(s)=-\infty$. Direct computation yields: $f_6(0)=f_6'(0)=f_6''(0)=f_6'''(s)=0$. Thus s=0 is a zero of $f_6(s)$ with multiplicity four. By lemma 1, there can exist at most one additional zero, whose multiplicity then is one. However, $f_6''''(0)=2(\alpha-1)\rho\delta(\delta-\rho)^2(\delta-\rho+\alpha\rho)/\alpha^2<0$, implying by Taylor's theorem that $f_6(s)$ must be negative for small s>0. But then we have shown that $f_6(s)<0$ $\forall s>0$. This implies that $\partial\Gamma_1/\partial m>0$ and $\partial s/\partial m>0$.

iii) Next we study $\partial s/\partial\rho|_{m=0}$. From (A1)

$$\partial\Gamma_1/\partial\rho|_{m=0}=e^{\rho s/\alpha}(e^{\delta s}-1)(px-w)\{e^{\delta s/\alpha+\rho s}[\alpha(\rho s-1)+\delta s-\rho s]+\alpha e^{\rho s/\alpha}\}/[\alpha p(e^{\delta s/\alpha+\rho s}-e^{\rho s/\alpha})^2].$$

The denominator of $\partial\Gamma_1/\partial\rho|_{m=0}$ is clearly positive. The term $e^{\rho s/\alpha}(e^{\delta s}-1)(px-w)$ in the numerator is positive. Dividing the term $e^{\delta s/\alpha+\rho s}[\alpha(\rho s-1)+\delta s-\rho s]+\alpha e^{\rho s/\alpha}$ by $e^{\rho s/\alpha}$ yields

$e^{(\delta s-\rho s)/\alpha+\rho s}[\alpha(\rho s-1) \quad +\delta s-\rho s]+\alpha\equiv u_8$. Since $u_8|_{\delta=\rho}=\alpha[e^{\rho s}(\rho s-1)+1]>0$ and $\partial u_8/\partial\delta=s^2e^{(\delta s-rs)/\alpha+rs}(\alpha\rho+\delta-\rho)/\alpha>0$, the numerator of $\partial\Gamma_1/\partial\rho|_{m=0}$ is positive. Thus $\partial\Gamma_1/\partial\rho|_{m=0}>0$.

(iv) For studying $\partial s/\partial w$, differentiation yields

$$\partial\Gamma_1/\partial w=\{\delta e^{\delta s/\alpha+\rho s}+e^{\rho s/\alpha}[\rho(\alpha-1)-e^{\delta s}(\alpha\rho+\delta-\rho)]\}/[p(1-\alpha)(e^{\delta s/\alpha+\rho s}-e^{\rho s/\alpha})].$$

The denominator of $\partial\Gamma_1/\partial w$ is positive. Define the numerator as $u_3=e^{s\rho/\alpha}z_3(s)$, where $z_3(s)=\rho(\alpha-1)+\delta e^{(\delta s-\rho s)/\alpha+\rho s}-e^{\delta s}(\alpha\rho+\delta-\rho)$. This yields $z_3(0)=0$. Since $(\delta-\rho)/\alpha+\rho>\delta$, $z_3(s)\to\infty$ as $s\to\infty$. In addition, $z_3'(0)=-\delta[\alpha^2\rho+\alpha(\delta-2\rho)-\delta+\rho]/\alpha$ implying that $z_3'(0)|_{\delta=\rho}=\rho^2(1-\alpha)>0$ and $\partial[\alpha^2\rho+\alpha(\delta-2\rho)-\delta+\rho]/\partial\delta=\alpha-1<0$. Thus $z_3'(0)>0\ \forall\ \delta\geq\rho$. By lemma 1 we then obtain that $z_3(s)$ and u_3 are positive $\forall s>0$. Thus $\partial\Gamma_1/\partial w>0$ and $\partial s/\partial w>0$.

v) Next we study $\partial s/\partial p$. Differentiation and some rearrangement yields

$$\partial\Gamma_1/\partial p=e^{-\rho s}\Big\{e^{\delta s/\alpha+\rho s}[m\rho e^{\delta s}(\alpha-1)+\delta e^{\rho s}(m-\rho w)-m(\alpha\rho+\delta-\rho)]-e^{\rho s/\alpha}\{e^{\delta s}[e^{\rho s}(m-\rho w)(\alpha\rho+\delta-\rho)-\delta m]+\rho e^{\rho s}(\alpha-1)(\rho w-m)]\}\Big\}/[p^2\rho(1-\alpha)(e^{\delta s/\alpha+\rho s}-e^{\rho s/\alpha})]= \qquad (A7)$$

$$-\partial\Gamma_1/\partial w(w/p)-\partial\Gamma_1/\partial m(m/p). \qquad (A8)$$

Since $\partial\Gamma_1/\partial w$ and $\partial\Gamma_1/\partial m$ are positive, $\partial\Gamma_1/\partial p<0$ if w and p are not both zero.

vi) The claim that m=0 and w=0 implies $\partial s/\partial p=0$ follows immediately from (A8).

vii) (A1) and (A8) yield $\partial\Gamma_1/\partial p|_{w=\sigma_1 p}=\partial\Gamma_1/\partial p+\sigma_1\partial\Gamma_1/\partial w=-\partial\Gamma_1/\partial m(m/p)$. Since we have shown that $\partial\Gamma_1/\partial m>0$, the proof follows immediately.

viii) Differentiation and (A8) yields:

$$\partial\Gamma_1/\partial p|_{\substack{w\equiv g_1 p\\ m\equiv\delta_2 p}}=\partial\Gamma_1/\partial p+\sigma_1\partial\Gamma_1/\partial w+\sigma_2\partial\Gamma_1/\partial m=0.$$

ix) To study $\partial s/\partial\alpha$, we define $k=\rho+(\delta-\rho)/\alpha$. Computing yields

$$\partial\Gamma_1/\partial\alpha=-(c_0/p)(1-e^{gs})/(1-\alpha)^2+\alpha(c_0/p)(\delta-\rho)se^{gs}/[(1-\alpha)\alpha^2]+\alpha(1-e^{gs})(c_0/p)(\delta-\rho)[1-kse^{-ks}/(1-e^{-ks})]/[(1-\alpha)k\alpha^2]=(c_0/p)f_4(s)/[(1-\alpha)(e^{ks}-1)],$$

where $f_4(s)=\delta/[k(1-\alpha)]-s(\delta-\rho)/\alpha-\delta(e^{ks}+e^{gs})/[k(1-\alpha)]+\{\delta/[k(1-\alpha)]-s(\delta-\rho)/\alpha\}e^{(g+k)s}$. We obtain $f_4(0)=f'_4(0)=f_4''(0)=f_4'''(0)=0$ and $f_4''''(0)=-2(1-\alpha)\delta(\delta-\rho)^2k/\alpha^3<0$. In addition, $f_4(s)\to-\infty$ when $s\to\pm\infty$. By lemma 1 we know that the sum of multiplicities cannot exceed 5. Of these $s=0$, is a zero with multiplicity of four. The fact that $f_4''''(0)<0$ and $f_4(s)\to-\infty$ when $s\to-\infty$ implies that there must be one zero with $s<0$. Thus $f_4(s)<0$ $\forall s>0$ and $\partial s/\partial\alpha<0$.■

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