COMMUNICATIONES INSTITUTI FORESTALIS FENNIAE 134

MIXED LINEAR MODELS FOR ANALYZING AND PREDICTING STEM FORM VARIATION OF SCOTS PINE

JUHA LAPPI

SELOSTE

MÄNNYN RUNKOMUODON ANALYSOINTI JA ENNUSTAMINEN LINEAARISTEN SEKAMALLIEN AVULLA

HELSINKI 1986

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Cover (front & back): Scots pine (*Pinus sylvestris* L.) is the most important tree species in Finland. Pine dominated forest covers about 60 per cent of forest land and its total volume is nearly 700 mil. cu.m. The front cover shows a young Scots pine and the back cover a 30-metre-high, 140-year-old tree.

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JUHA LAPPI

MIXED LINEAR MODELS FOR ANALYZING AND PREDICTING STEM FORM VARIATION OF SCOTS PINE

Academic dissertation to be presented, with the assent of the Faculty of Social Science of the University of Helsinki, for public discussion in Auditorium Porthania III, Hallituskatu 11, on May 27th, 1986, at 12 noon.

SELOSTE

MÄNNYN RUNKOMUODON VAIHTELUN ANALYSOINTI JA ENNUSTAMINEN LINEAARISTEN SEKAMALLIEN AVULLA

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The model presented provides a compact general description of how the stem form of Scots pine varies as a multidimensional object. Stem dimensions are determined in polar coordinates, and tree size is defined as a weighted mean of logarithmic dimensions. Logarithmic dimensions are analyzed by a linear model in which the size and relative size of a tree are fixed explanatory variables. The random variation in stem dimensions is partitioned into variation between stands and variation within stands. The principal components of the between-stand and within-stand covariance matrices are used to give a more economical description of the random variation.

In applications, tree sizes are the parameters to be estimated, and parameters of the stem form model are known variables. After some approximative derivations, the model can be presented in the standard form. The stem curve can be calibrated for a given stand by estimating the random stand effects by means of the first few principal components of the between-stand covariance matrix. The model can be applied if any stem dimensions are measured. With the usual measurements the model is as good as normal regression equations. If the height and diameter at breast height are measured for a single tree in a stand, the between-stand error variance is already reduced by 70 % when the volumes of tally trees (i.e., trees for which only diameter at breast height has been measured) are estimated. Error variances for diameters and for stem volumes are also obtained. Hence the model can be used to study theoretically different measurement strategies, e.g., optimal heights for diameter measurements and optimal measurement combinations for sample trees. The model can also be applied in timber assortment problems. Measurement errors can be corrected to some extent by incorporating variances of the measurement errors into the model.

The stem form model is based on the standard theory of mixed linear models. Because the most compact prediction formulas for a mixed linear multivariate model apparently are not available elsewhere, they are derived in this study.

Tutkimuksessa esitetään yleinen männyn runkomuodon vaihtelun malli. Mallissa runkomuoto kuvataan läpimittojen moniulotteisena vektorina. Läpimitat ilmaistaan napakoordinaatistossa. Puun koko määritellään logaritmisten läpimittojen painotettuna keskiarvona. Logaritmiset läpimitat kuvataan yksinkertaisella lineaarisella mallilla, missä puun koko ja suhteellinen koko ovat kiinteitä selittäjiä. Satunnaisten metsikkö- ja puutekijöiden avulla runkomuodon satunnaisvaihtelu jaetaan metsiköiden väliseen ja metsikön sisäiseen vaihteluun. Metsiköiden välisen ja metsikön sisäisen vaihtelun kovarianssimatriiseja analysoidaan pääkomponenttien avulla.

Mallia sovellettaessa puun koko tulkitaan satunnaiseksi tai kiinteäksi parametriksi, ja analyysivaiheen parametrit ovat muuttujia. Malli kalibroidaan metsikkökohtaisesti estimoimalla satunnaiset metsikkötekijät pääkomponenttien avulla. Mallia voidaan soveltaa olipa metsikön puista mitattu mitä tahansa dimensioita. Yleisillä mittauskombinaatioilla malli antaa yhtä hyviä tuloksia kuin normaalit regressiomallit. Lukupuiden tilavuusestimaattien metsikkövirhe pienenee jo 70 prosentilla, kun yhdestä koepuusta on mitattu pituus ja rinnankorkeusläpimitta. Läpimittojen ja tilavuuksien virhevariansseille saadaan estimaatit. Näiden avulla voidaan tutkia teoreettisesti mittausstrategioita, esim. etsittäessä optimaalista läpimitan mittauskorkeutta tai optimaalista koepuiden mittauskombinaatiota. Virhevariansseja käytetään myös puutavaralajiongelmissa. Mittausvirheiden varianssit voidaan ottaa estimoinnissa huomioon.

Tutkimuksessa sovelletaan yleistä lineaaristen sekamallien teoriaa. Yleinen lineaarinen ennustin ja sen virhevarianssi johdetaan sekamalleissa helpommin soveltuvaan muotoon.

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APPENDIX

PREFACE

Dr Pekka Kilkki suggested that I begin this study. His critical questions and comments have guided my work, and he also provided me with his data on stem form. Later, Dr Jouko Laasasenaho allowed me to use his exellent data. Lic. Timo Pekkonen prepared a computer program by which I could make comparisons with his calibration system for volume equations. Pekkonen and Laasasenaho also provided me with their data collected for volume estimation of standing trees.

Mr Jamel Boucelham, M.S., Dr Jouko Laasasenaho, Dr Seppo Mustonen, Dr Hannu Niemi, Mr Risto Ojansuu, M.F., Lic. Timo Pekkonen and Dr Markku Rahiala have made useful comments during the course of the study. The language was checked by Dr Joann von Weissenberg and Mr Jamel Boucelham. Mr Pekka Voipio helped in the plotting of figures.

The generous attitude of Dr Erkki Lähde made it possible for me to complete this study in forest inventory while holding a permanent appointment in the Department of Silviculture of The Finnish Forest Research Institute.

I wish to thank all who have helped me in my work.

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Suonenjoki, February 1986

Juha Lappi

1. INTRODUCTION

11. Background

In forestry, modelling of stem form is of prime importance for both practical and theoretical reasons. For forest mensuration, stem form is of interest in the determination of the volume and value of the whole stem or a part of it. In this study, a general statistical model will be presented, which is used for analyzing variation in stem form and for estimation problems. Biological theories are of less interest here, as are empirical studies that relate some specific variables to each other or use geometrical solids to describe different parts of the stem. Only a small fraction of the vast literature on stem form (reviewed by Sterba 1980) is directly related to this work.

Fries and Matérn (1966) and later Liu and Keister (1978) used multivariate statistical methods (principal component analysis) to describe stem form and its variation. In this study stem form is also described as a multidimensional variable rather than a relation between diameter and height. According to Sloboda (1977a), stem dimensions are expressed in polar coordinates.

Sloboda (1977b) and Lahtinen and Laasasenaho (1979), for example, have used splines to interpolate the stem curve between measured points. Here splines are used to interpolate stem curves and various parameters. Covariances are interpolated with two-dimensional splines.

In biology, problems of shape and size have a long research tradition. Here a new approach is sought for these allometric problems. The papers of Mosimann (1970) and Sprent (1972) provided the key insights into the definition of size.

This study is based on the data Laasasenaho (1982) used to formulate his taper curve and volume functions for pine. His results will be the main reference when models derived from the general model for stem form are compared with special functions.

The starting points for this study were the papers of Kilkki et al. (1978), and Kilkki & Varmola (1981), in which a multi-equation model introduced the idea of a single general model that can be used when any combination of measurements is available. Kilkki's study of sample tree selection (Kilkki 1983) has also brought about the decomposition of total variation into variation between forest stands and variation within stands. In this respect, the study of Pekkonen (1983) also guided the formulation of the problem.

12. Purpose of the study

The goal of this study is to provide a unifying framework for stem curve problems where the essential features of stem form variation can be analyzed and the analysis is then directly applicable for various purposes.

In this analysis, the size of a tree is first defined as a fixed artificial variable. Simple linear models are then used to describe expected (average) stem dimensions as functions of size. Together these functions describe the dependence of the stem form on the size. Deviations from the expected stem curve are partitioned into two random components: stand effects and tree effects. Estimation and interpretation of both variances and covariances of the stand effects, as well as the tree effects, are the main tasks in analysis of the random variation of the stem form.

In applications, the model should satisfy the following requirements:

a) A stem curve can be predicted if any dimensions of the tree are measured.
b) Predicted stem curves pass through the measured points, if there are no measurement errors.
c) Estimates of the error variances are obtained.
d) The results are unbiased for the main population without any stand information.
e) The general model can be efficiently calibrated for a stand if any dimension combinations of different trees are measured.
f) Knowledge of possible measurement errors is taken into account in the prediction.

Several existing ideas can be synthesized

in a new way, so that the variation in stem form is analyzed in a theoretically compact and informative model satisfying the above requirements. The model is kept as simple as possible; it is aimed to be a starting point for more sophisticated models.

The purpose of this report is to describe the ideas and derivation of the model and to demonstrate its applicability. The study is based on the standard theory of linear models. However, to be more applicable in the multivariate case with both random and fixed parameters, the available prediction formulas need to be developed further. Mathematical derivations and details are presented in appendices; readers interested only in the general ideas of the model may omit these details. Appendix A is recommended for readers not familiar with the basic concepts of mixed linear models. Preliminary ideas of this study were presented in Lappi (1983); more recent results were summarized in Lappi (1985).

13. Computational aspects

All calculations are based on standard matrix algebra and on spline interpolation. Programs were written in Fortran-77 and run in the VAX-11/780 computer of the Finnish Forest Research Institute. IMSL-subroutines (IMSL ... 1982) were used whenever possible and applicable. In order to guarantee sufficient numerical accuracy, computing was done in double precision and some attention was also paid to the scaling of variables.

Interpolations were done by cubic splines, i.e., by cubic polynomials joined so that the first and second derivatives are continuous. The stem curves were interpolated using spline-subroutines based on the study of Lahtinen and Laasasenaho (1979). The parameters were interpolated by an IMSL-routine, which uses the "not-a-knot" condition requiring that the third derivative be continuous at the second and penultimate knots.

Direct application of matrix operations would in some cases lead to inefficient computations, and therefore some attention is paid to the usage of special matrix structures. Explicit presentation of special matrix structures may also help to understand the link between general theory and the special case.

14. Terminology and notation

In this study, nonelementary statistics is applied in a specific problem of forestry. This report is intended both for foresters with limited background in statistics and for statisticians without any knowledge of forestry. Readers may thus have conflicting wishes for the level and details of the presentation. To make the following text easier to understand, some terminology is first explained.

Forest stands and pine trees within stands are considered here. A stand is a forest area that is relatively homogenous both with respect to the site and structure of the forest. In statistics these kinds of groups are normally called classes. Thus in this study random (between-)class effects (or just 'random effects') are called random (between-)stand effects. In the context of the general linear model, random effects and random parameters are used as synonyms. In standard statistical writing, random tree effects (within-stand effects) correspond to random errors (= residual erros = within-class effects).

If $\hat{x}$ is said to be an unbiased estimator of x, this can have three different meanings depending on the statistical characters of x and $\hat{x}$. If x is a fixed parameter and $\hat{x}$ is computed from the observed values of random variables, then $E(\hat{x})=x$. If x is a random variable and $\hat{x}$ is fixed, then $E(x)=\hat{x}$. Estimator $\hat{x}$ can be fixed either because it is computed from observed values of fixed variables or from conditionally fixed random variables (expectation is taken with respect to the conditional distribution). If both x and $\hat{x}$ are random, then $\hat{x}$ is unbiased for x if $E(x)=E(\hat{x})$. If x is random, $\hat{x}$ is usually called a predictor instead of an estimator.

In this report formulas are numbered according to sections. Vectors and matrices are printed in boldface; lower case letters are used for vectors and capital letters for matrices. Vectors are column vectors by default. Many symbols have only a local meaning defined as they are used, sometimes the same letters may be needed for different purposes in different parts of the text. The symbols are defined as they appear, but general symbols are collected to Appendix D for easy reference.

2. DATA

Three sets of data were used in this study: a data set collected by Laasasenaho (the primary data), a second set collected by Kilkki and Varmola and a third one collected by Pekkonen and Laasasenaho. Detailed descriptions of these respective sets are given by Laasasenaho (1982), Kilkki and Varmola (1981), and Pekkonen and Laasasenaho (1985).

Trees in the data of Laasasenaho were collected from sample plots selected from the tracts of the National Forest Inventory and covering the whole of Finland (Fig. 1). Sample trees within each sample plot were selected using relascope with factor 2 (i.e., sampling probabilities are proportional to the square of the diameter at 1.3 m). However, at most 5 trees per plot have been accepted. Each sample plot is supposed to represent a stand. Data set consists of 956 sample plots with 2326 trees. Sample plots with one sampled tree do not contain any information about the within-stand variation and the effect of the relative size of the trees, which are both constituents of the model. Therefore all sample plots with one sampled tree (340 plots) were excluded from the data in the analysis stage. The data thus reduced consisted of 616 plots and 1986 trees.

On the sample trees of Laasasenaho, the diameter was measured at relative heights of 1, 2.5, 5, 7.5, 10, 15, 20, 30, 40, 50, 60, 70, 80 and 90 percent, and at heights of 1.3 and 6 meters. The crown height (i.e., the height at which the live crown begins) and the height of the uppermost root collar were also measured. The data of Laasasenaho were used to estimate the model parameters and also to test the performance of the model in different applications.

Standwise calibration of the stem curve was also tested with the data of Kilkki and Varmola and with that of Pekkonen and Laasasenaho. The data of Kilkki and Varmola consist of 492 trees in 29 subjectively chosen stands in southern and central Finland. From each stand, 5—20 trees were measured. Measured trees were selected systematically so that there would be about equal numbers of trees in different breast-height diameter and height classes in the whole data set. The above-mentioned measurements are also available in the data of Kilkki and Varmola, except for the height of the uppermost root collar.

From the data of Pekkonen and Laasasenaho, 2418 Scots pines from 26 pine-dominated stands were used. In their data the diameters were measured at 1.3 and 6 meters and at one-meter intervals.

It is assumed for all three data sets that cubic splines going through the measurements give the stem curves precisely. Thus volumes are also assumed to be 'measured'.

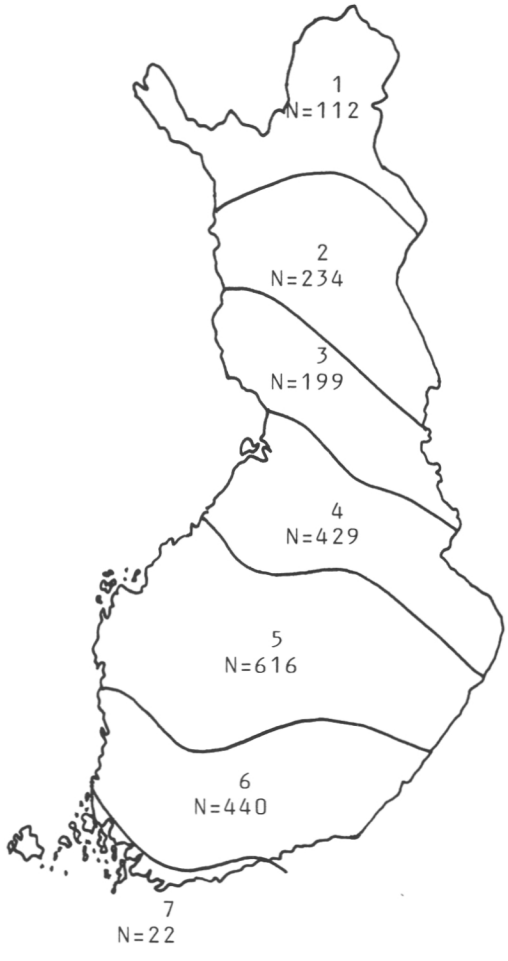

Fig. 1. Number of trees in the data of Laasasenaho according to climatic regions; trees growing on wastelands (N=274) are not included.

3. SHAPE, SIZE AND DIMENSIONS OF A TREE

31. Dimensions

When studying the shape and size of an organism quantitatively, one must first decide how its dimensions are to be defined and measured. Generally, dimensions are defined as distances between biologically analogous points. With trees, however, definitions of dimensions are not as evident as in most allometric studies. Height, H, (i.e., the distance between ground level and the top of the tree) is in any case a basic dimension of a tree stem. Ignoring the noncircularity of tree stems, other dimensions of a stem can be defined in terms of diameters. The only problem is to determine which diameters of different trees represent the same dimensions.

In this study a polar coordinate system (Fig. 2) is used to define the dimensions of a tree stem (as did Sloboda 1977a). Relative-height diameters, which have been more popular (see, e.g., Cajanus 1911), could not be used in this study, because the estimation is set to work when any dimensions are measured, e.g., even if height is not measured.

In a polar coordinate system the dimensions can be either rays or diameters corresponding to different angles. There is a simple one-to-one relationship between the rays and diameters. Let R(u) be the ray and D(u) the diameter at angle u. Then

$$D(u) = \cos(u)R(u), \text{ or} \tag{31.1}$$

$$R(u) = D(u)/\cos(u). \tag{31.2}$$

When $u = 90°$, R(u) cannot be expressed in terms of D(u). If rays at different angles are used as basic variables, then the height is just one ray among others.

Geometric properties of plane figures are independent of the scaling of the coordinates. Thus scaling can be chosen on practical grounds and has no real effect on the properties of the model. In order to get roughly circular forms with understandable angles, diameters are expressed in centimeters and heights in meters. This scaling is also reasonable for the numerical accuracy of the computations.

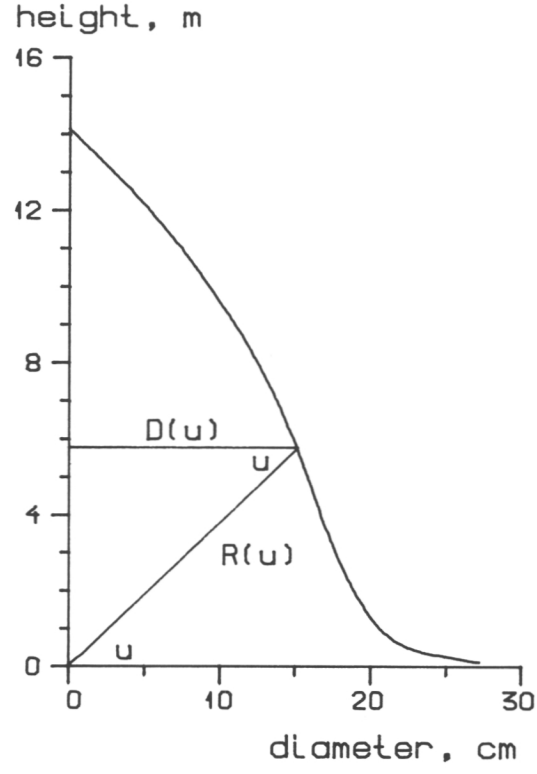

Fig. 2. The polar coordinate system where the stem dimension for angle u is either ray R(u) or diameter D(u).

As noted, height is naturally included among the dimensions. For angles other than 90° one can equally well use either rays or diameters. Within this artificial scaling, the rays are not in any physical measurement units, so the results will be expressed in terms of diameters. As height is not any diameter at some angle, there exists a continuity gap between the uppermost diameter and the height. Thus the interpolations are, in fact, based on the rays, even though this report is written in terms of diameters.

In the data used here diameters were measured at given heights. Diameters at different angles were computed with splines. A sufficiently good presentation of the stems seems to be attainable with 12 different angles (plus 90° for the height) which are called 'knot angles'. The knot angles were selected with a heuristic 'inverse spline' criterion.

Let g be any function having an inverse g^{-1}. Then $g^{-1}[g(x)] = x$ for all x. Thus when moving from x to g(x), no information about x is lost, if the inverse of g exists and is known. This idea is extended to splines, which are functions defined by the knot points. The problem is to find angles in the polar coordinate system so that the information contained in the original measurements prevails. This is the case if, using an 'inverse spline', we can move to the original points. The original splines were first defined for each tree with the available measurements. Then the knot points for the inverse splines were interpolated in accordance with the polar coordinate system. The selected knot angles (Fig. 3), and the mean and standard deviation of the corresponding relative heights (H(u)/H) were:

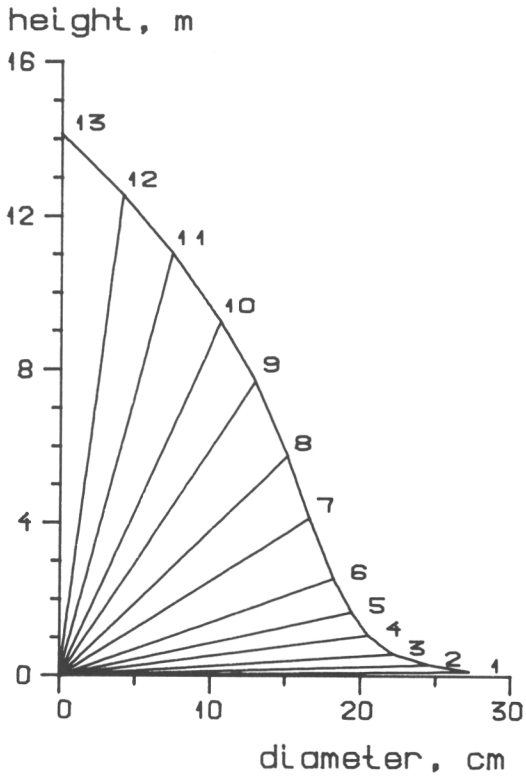

Fig. 3. Knot angles used in the analysis.

	u, deg.	H(u)/H, %	
		mean	sd
1	0.25	0.87	0.22
2	0.7	2.2	0.54
3	1.5	4.3	1.0
4	3	7.9	1.9
5	5	12.5	2.9
6	8	18.8	4.3
7	14	30.3	6.2
8	21	41.9	7.3
9	31	55.4	7.2
10	41	65.9	6.3
11	56	78.1	4.5
12	72	88.7	2.6

When the inverse splines were used for computing the diameters at relative heights, biases for all relative heights were less than 0.7 mm in absolute value and standard deviations less than 2.7 mm.

In the analysis stage, finite-dimensional multivariate models are formulated in which the knot angles determine the dimensions. The true models, however, are assumed to be continuous with respect to the measurement angle. In the applications the model quantities are interpolated for all angles. Thus, already in the analysis stage the parameters are written as functions of angle rather than using indices.

To simplify the notation, we write u = 1,..,12 for the knot angles, u = 13 for the height and u = 14 for the crown height. Thus for a parameter v, for example, v(3) is equivalent to v(1.5°), and v(14) is the parameter for the crown height.

32. Shape and size

This study analyzes the relationship between the shape and size of a tree stem. Two stems have, by definition, the same shape if

$$D_2(u) = cD_1(u) \text{ for all } u. \qquad (32.1)$$

Suppose first that all stems have the same shape. Then any weighted mean of stem dimensions can be defined to be the size of the stem. Thereafter the stem of any tree i can be presented as:

$$D_i(u) = S_i F(u),$$

where S_i is the size of the stem and the function F describes the common shape of stems. Every stem is determined precisely when any dimension is known.

As not all stems have the same shape, the model must be modified. First, the dimensions may not increase in proportion to the size but to some function of it. Second, there exists random variation, which, one can assume, enters into the model multiplicatively, i.e., is roughly proportional to size. Thus, any stem i can be presented as:

$$D_i(u) = G(u,S_i)E_i(u), \tag{32.2}$$

where G expresses how the average shape depends on the size and $E_i(u)$ is a random error independent of size. Even if not explicitily indicated, G may be a function of other tree and stand variables as well.

Taking logarithms, we obtain:

$$\ln[D_i(u)] = \ln[G(u,S_i)] + \ln[E_i(u)]. \tag{32.3}$$

Let us then define:

$$d(u) = \ln[D(u)]$$

$$s = \ln(S)$$

$$g(u,s) = \ln[G(u,S)] \tag{32.4}$$

$$\epsilon(u) = \ln[E(u)].$$

Then (32.3) can be written as:

$$d_i(u) = g(u,s_i) + \epsilon_i(u). \tag{32.5}$$

The presented formulation has no concrete meaning before the size is defined operationally. In addition, the statistical nature of the size needs special consideration. In the following, S and s are called 'arithmetic size' and 'logarithmic size', respectively.

33. Operational definition of size

There is no universal definition of size of an organism, although it is generally understood as an aggregate quantity combining different variables associated with it. When the whole interest lies in geometric aspects, the relevant variables are distances, and the arithmetic size (S) can be expressed in the same physical units as the dimensions. Thus, size S (or s) must be defined so that if all dimensions are multiplied by α, so is the size S; and the logarithmic size s is thus increased by $\ln(\alpha)$.

Principal components have been used to define and study the variation in shape and size (e.g., Fries and Matérn 1966). Here the first principal component is interpreted as the size component (see, e.g., Sprent 1972, and Morrison 1976). If different individuals have approximately the same shapes but different sizes, then the greatest variation in the vectors of measured dimensions is in the direction of size. Using logarithms, the variation is measured in relative units.

The sample covariance matrix of log-diameters was computed without paying any attention to the stand structure. Then the first characteristic vector of the matrix was:

$$(.286, .280, .279, .279, .279, .280, .284, .284, .281, .278, .271, .264, .261)' \tag{33.1}$$

The characteristic vectors are scaled so that the sum of the squared elements is one. Because the arithmetic size S is to be multiplied by the same constant by which every dimension is multiplied, coefficients in the definition of the logarithmic size s must be scaled to add up to one. Thus the elements of the vector (33.1) must be divided by the sum of elements (3.604) yielding a vector $\mathbf{w}' = [w(1),..., w(13)]$. The size is then defined to be:

$$\begin{aligned} s &= \sum_{u=1}^{13} w(u)d(u) \\ &= 0.0793d(1) + 0.0776d(2) + 0.0774d(3) + \\ &\quad 0.0775d(4) + 0.0773d(5) + 0.0778d(6) + \\ &\quad 0.0787d(7) + 0.0787d(8) + 0.0780d(9) + \\ &\quad 0.0770d(10) + 0.0753d(11) + 0.0731d(12) + \\ &\quad 0.0723d(13) \end{aligned} \tag{33.2}$$

Since the coefficients are approximately equal, an alternative definition of the arithmetic size would be:

$$S = [\prod_{u=1}^{13} D(u)]^{1/13},$$

leading to the logarithmic size

$$s = [\sum_{u=1}^{13} d(u)]/13.$$

If the coefficients of the first principal component of a multinormal distribution of

logarithmic dimensions are all equal, the growth of the dimensions is isometric, or in constant proportion to the increasing size (Mosimann 1970). As it is not assumed here that the measured diameters come from a single multinormal distribution, the standard statistical interpretation and testing procedures cannot be used. The definition of size used here seems to give a good operational basis for the analysis of stem form variation, but it is not necessarily the 'correct' one.

At an earlier stage of this study, the arithmetic size of the stem was defined as the cubic root of the volume. From a practical point of view, the results were as good as those presented here. This definition would, however, lead to a slight logical inconsistency. When moving from volume to diameters and then back to volume, we do not necessarily get the same volume we started with. Defining the size by formula (33.2), a logically consistent system can be achieved. In practice, these two possible definitions of size are closely related; volume (in liters) can be predicted by the equation:

$$V = 0.05962\ S^{3.042}. \tag{33.3}$$

The value of R^2 for the above regression was 0.995. The regression equation for the logarithms was:

$$\ln(V) = 3.042s - 9.728. \tag{33.4}$$

The logarithmic regression showed that the standard deviation of relative errors was 4 %.

Size is defined as a weighted mean of diameters, and each diameter is supposed to have a random component. Thus size, as the sum of random variables, is also a random variable. Nevertheless, it will be considered as fixed, and this is claimed to be logically consistent.

First, the main task in the analysis stage is to model the variation in stem form. It is then quite natural to take the size as given, i.e., determined outside the system that determines the shape of stem. In the application stage, the situation can be different, as will be discussed later. Second, even if the size is formally a random variable, it behaves like a fixed variable, if the parameters of the model meet certain constraints. These constraints will be discussed later in more detail.

The primary reason for insisting on the nonrandomness of the size is, of course, that the statistical treatment becomes simpler and more straightforward. The characterization and use of size is perhaps the clearest point at which our approach diverges from the traditional allometric studies.

4. STEM FORM MODEL

41. Model specification

After defining the size of a stem, let us study the dependence of the expected stem form on size and the behavior of the random variation. Introduce the stand structure into the model (32.5) through the index k:

$$d_{ki}(u) = g(u,s_{ki}) + \epsilon_{ki}(u). \tag{41.1}$$

The first assumption will be that $E[\epsilon_{ki}(u)] = 0$. Thereafter ordinary least squares regression can be used to search for an appropriate model, even if alternative models cannot be compared exactly with F-values. A good starting point in the model development is:

$$g(u,s) = s + f(u), \tag{41.2}$$

where f(u) describes the average stem form. In this model the expected shape of a stem is independent of the size.

As noted in section 32, g can, in addition to s and u, be a function of any stand and tree variables. If g is defined in terms of size variables only, the following model seemed to describe the stem form of tree i in stand k:

$$d_{ki}(u) = a_0(u) + a_1(u)s_{ki} + a_2(u)s_{ki}^2 + a_3(u)\bar{s}_{k.} + v_k(u) + e_{ki}(u), \tag{41.3}$$

where d is logarithmic diameter,
s is logarithmic size,
$\bar{s}_{k.}$ is the average size in stand k,
v_k is a random stand effect with zero mean and fixed variance,
e_{ki} is a random tree effect with zero mean and fixed variance, and a_0, a_1, a_2 and a_3 are fixed parameters.

The random effects v and e are assumed to be uncorrelated with each other, and the e's are also assumed to be uncorrelated for different trees. For a given stand the v's of different angles are correlated, and for a given tree the e's of different angles are correlated. In the analysis stage, the distributions of v and e need not be specified. In applications some derivations are valid only if v and e, for different angles u, follow multinormal distributions. Normality assumptions will be always stated explicitly.

The model contains both ordinary fixed parameters (a's) and random parameters (v's), thus it is a special case of mixed linear models (see Appendix A). Note that the model contains both a fixed stand effect $a_3(u)\bar{s}_{k.}$ and a random stand effect $v_k(u)$. The model (41.3) can be written in the following equivalent form:

$$d_{ki}(u) = a_0(u) + a_1^*(u)s_{ki} + a_2(u)s_{ki}^2 - a_3(u)(s_{ki} - \bar{s}_{k.}) + v_k(u) + e_{ki}(u), \tag{41.4}$$

where

$$a_1^*(u) = a_1(u) + a_3(u). \tag{41.5}$$

The term $s_{ki} - \bar{s}_{k.}$ represents the relative size of a tree in comparison to other trees in the same stand. The competitive status of a tree is taken into account through this term.

The model is a three-level model. Its fixed part defines the 'population stem curve'; the fixed part plus the v-effects define the 'stand stem curve', and the whole model defines the 'tree stem curve'. Since the v-effects are taken to be random, we can study the variation of the stand stem curve using their variances and covariances. The effect of the average size in (41.3) could also be included in the 'stand stem curve', similarly the effect of the relative size in (41.4) could be included in the 'tree stem curve'.

The fixed part of the model was first fitted by ordinary least squares. The expected values of the obtained residuals are zero under the assumptions of the model, even if the residuals within the same stands are correlated. Then, in order to study systematic deviations from the model, the residuals were tabulated by different variables describing stand and tree characteristics.

The most important differences seemed to be between climatic regions. These differences may be explained, in addition to the pure climatic factors, by differences in site quality and silvicultural history of the stands. These characteristics have the same systematic geographic trends as the climatic factors. Consequently, the coefficients of the fixed effects were estimated separately for each climatic region. The stands growing on sites classified as wastelands deviated so

much from the others that they were treated as a separate class. To simplify the terminology, hereafter this class is also called a region.

The random part of the model was assumed to be the same for all climatic regions, so the model parameters were estimated simultaneously, i.e., technically as a single model. As the representativeness of the data is not very good with respect to climatic regions, a single model was also estimated for the whole country. The differences between the overall model and the regionalized model illustrate what happens when the model is made more specific.

It is logically consistent to assume that both the overall and regionalized models are correct simultaneously. The overall model applies when a stand is taken randomly from the population formed by all the stands in the country while in the regionalized model conditional inferences are made for a given region.

The age of the tree, the site class, and the density of the stand had also a noticeable effect on the stem form. In order to concentrate on the main allometric relations, these variables were not included in the model. So the model will be applicable even if the values of these variables are not known.

42. Fixed parameters

Intuitively, it is more natural to consider the fixed part of the model first, even if the variance-covariance components must be estimated before the fixed parameters. Estimation of the parameters of a mixed linear model is presented in general terms in Appendix A.2. A more detailed description of the estimation of the parameters in the present special case is given in Appendix B.2.

Briefly, there are two different ways of estimating the fixed parameters of the model. Model (41.3) or (41.4) has separate parameters for each angle u. Thus the parameters can also be estimated separately for each angle. The random stand effects and tree effects of different angles are, however, correlated. Therefore the parameters can be estimated more efficiently by estimating them simultaneously for all angles. This is known as estimation of seemingly unrelated regressions (see, e.g., Johnston 1972). With the computing capabilities available, the simultaneous system could be solved only for regions with a small number of stands in the regionalized model.

Table 1 shows $\hat{a}_0(u)$, $\hat{a}_1(u)$, $\hat{a}_2(u)$ and $\hat{a}_3(u)$ ($u=1,\ldots,14$) with their estimated standard errors for the overall model and for each region in the regionalized model. The sums of $w(u)a(u)$ over the knot angles are also given for later discussion; the w-coefficients are the coefficients in the definition of the size (33.2). The parameters for regions 1, 2, 3 and 7 were estimated using the multivariate model ('seemingly unrelated regressions'). These parameters were also estimated using the univariate model. The estimates and their standard errors were very close for both methods of estimation. The advantage of using the multivariate model would be even smaller for regions with more trees and stands.

The regional differences in the average stem curves are illustrated in Fig. 4. The

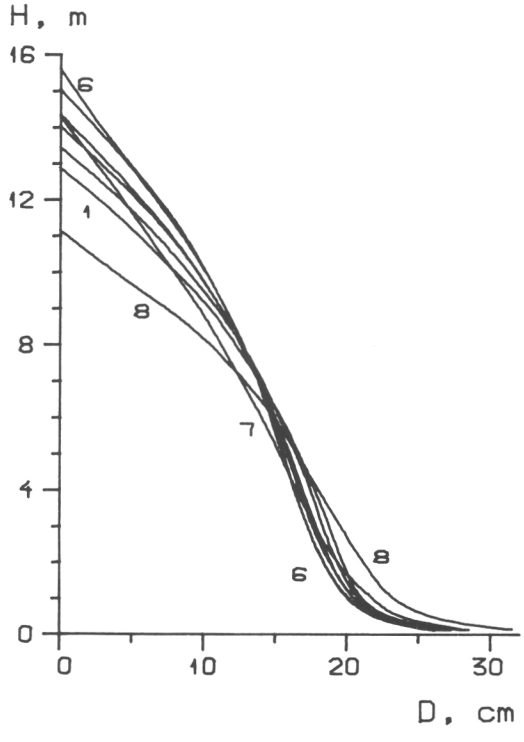

Fig. 4. Expected stem form for a tree with size s=2.7 (about 220 dm^3) for each region (see Fig. 1) according to the regionalized model when the relative size is zero. The most exceptional regions are indicated; region 1 is northern Lapland, region 6 is southern Finland, region 7 is the southwestern coast, and region 8 is the class of wastelands.

Table 1. Estimates of $a_0(u)$, $a_1(u)$, $a_2(u)$ and $a_3(u)$ (upper figures) with their estimated standard errors (lower figures) for the overall model (region=0) and for each region in the regionalized model; region 1 is northern Lapland, region 7 southwestern coast and 'region' 8 is the class of wastelands. The sums of w(u)a(u) over the basic angles (u=1,...,13) are also given, where w(u) is the coefficient of d(u) in the definition of size (33.2). Parameters for regions 1, 2, 3 and 7 have been estimated using the multivariate model.

Region	u 1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\sum_{u=1}^{13}$ w(u)a(u)
								a_0							
0	.784	.793	.746	.558	.521	.483	318	.065	—.313	—.644	—1.141	—1.798	—.630	—2.168	—.00438
	.043	.035	.033	.030	.028	.026	.021	.016	.021	.030	.042	.051	.057	.169	
1	1.320	.972	.620	.371	.547	.317	.079	—.215	—.588	—.741	—1.053	—1.502	—.292	.668	.00000
	.271	.226	.212	.188	.178	.160	.135	.111	.134	.191	.263	.315	.354	1.105	
2	.640	.636	.572	.373	.363	.356	.243	.100	—.261	—.491	—.920	—1.502	—.260	—1.429	.00000
	.096	.082	.077	.070	.066	.059	.048	.038	.048	.071	.099	.118	.133	.416	
3	1.049	1.008	.927	.756	.705	.621	.343	.058	—.436	—.848	—1.395	—2.033	—1.010	—2.645	.00000
	.157	.133	.126	.113	.107	.095	.078	.062	.077	.115	.161	.192	.217	.676	
4	.807	.774	.751	.507	.461	.438	.331	.072	—.328	—.640	—1.088	—1.758	—.553	—2.489	—.00240
	.084	.070	.066	.059	.056	.050	.042	.033	.041	.060	.082	.098	.110	.345	
5	.948	1.018	1.006	.831	.727	.637	.373	.043	—.422	—.851	—1.449	—2.165	—1.053	—4.248	—.00722
	.091	.076	.072	.064	.061	.055	.046	.035	.044	.065	.089	.107	.120	.375	
6	.592	.621	.543	.461	.431	.406	.331	.093	—.264	—.542	—1.002	—1.525	—.240	—2.743	.00443
	.133	.112	.106	.095	.090	.081	.067	.050	.065.	096.	133	.159	.178	.558	
7	—.369	—.156	—.047	—.413	—.342	—.146	—.127	.358	.549	.479	.353	—.555	.429	2.601	.00000
	.902	.760	.715	.638	.603	.538	.448	.363	.447	.647	.900	1.077	1.210	3.756	
8	.244	.280	.244	.195	.192	.246	.233	.036	—.082	—.245	—.496	—1.024	.147	—.089	.00174
	.091	.077	.073	.065	.062	.056	.046	.034	.044	.066	.092	.110	.123	.386	
								a_1							
0	.958	.897	.853	.933	.903	.862	.866	.939	1.054	1.120	1.194	1.230	1.276	1.396	1.00348
	.038	.031	.029	.025	.024	.022	.019	.015	.018	.026	.035	.042	.047	.143	
1	.291	.497	.641	.736	.577	.739	.904	1.175	1.442	1.489	1.552	1.515	1.537	.693	1.00000
	.209	.172	.160	.141	.134	.120	.104	.088	.103	.144	.195	.234	.262	.816	
2	.868	.846	.818	.929	.915	.881	.896	.919	1.098	1.142	1.223	1.246	1.263	.905	1.00000
	.083	.069	.064	.057	.054	.048	.041	.035	.041	.058	.078	.094	.105	.326	
3	.820	.739	.709	.753	.748	.729	.824	.952	1.155	1.286	1.383	1.407	1.578	1.208	1.00000
	.174	.146	.137	.122	.115	.103	.086	.071	.086	.124	.172	.206	.231	.720	
4	.928	.889	.832	.960	.930	.886	.859	.942	1.075	1.123	1.168	1.224	1.248	1.601	1.00205
	.078	.064	.060	.052	.050	.045	.039	.032	.038	.053	.071	.086	.096	.299	
5	.860	.740	.671	.747	.769	.776	.835	.964	1.130	1.262	1.393	1.461	1.551	2.566	1.00590
	.081	.067	.063	.056	.053	.048	.041	.032	.040	.057	.077	.092	.103	.323	
6	.993	.932	.948	.992	.974	.933	.872	.955	1.058	1.079	1.124	1.060	1.047	1.628	.99629
	.113	.094	.089	.079	.075	.068	.057	.043	.055	.080	.109	.131	.147	.459	
7	1.859	1.802	1.684	1.848	1.658	1.345	1.177	.655	.249	.115	—.059	.166	.365	—2.033	1.00000
	.589	.493	.463	.411	.388	.347	.292	.240	.292	.417	.576	.690	.774	2.408	
8	1.337	1.292	1.224	1.154	1.059	.932	.841	.896	.886	.880	.818	.779	.846	.846	.99810
	.104	.087	.082	.072	.069	.062	.052	.040	.051	.074	.100	.120	.134	.420	

Table 1 contd.

Region	u 1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\sum_{u=1}^{13} w(u)a(u)$
								a_2							
0	.034	.043	.052	.033	.038	.046	.042	.019	—.018	—.044	—.073	—.093	—.108	—.162	—.00077
	.007	.006	.005	.005	.004	.004	.003	.003	.003	.005	.006	.008	.009	.026	
1	.166	.127	.102	.077	.109	.075	.034	—.030	—.100	—.124	—.148	—.152	—.163	—.007	.00000
	.039	.032	.030	.027	.025	.023	.020	.017	.020	.027	.037	.044	.049	.154	
2	.052	.051	.056	.030	.032	.037	.031	.020	—.029	—.047	—.072	—.085	—.092	—.016	.00000
	.015	.013	.012	.010	.010	.009	.008	.006	.008	.011	.014	.017	.019	.060	
3	.082	.084	.087	.071	.070	.073	.052	.016	—.043	—.084	—.122	—.137	—.174	—.191	.00000
	.028	.024	.022	.020	.019	.017	.014	.011	.014	.020	.028	.034	.038	.117	
4	.040	.047	.060	.030	.032	.041	.046	.022	—.022	—.046	—.071	—.095	—.109	—.216	—.00048
	.014	.012	.011	.010	.009	.008	.007	.006	.007	.010	.013	.016	.018	.055	
5	.050	.071	.086	.069	.063	.064	.050	.017	—.027	—.068	—.112	—.140	—.166	—.417	—.00128
	.015	.012	.012	.010	.010	.009	.008	.006	.007	.010	.014	.017	.019	.059	
6	.029	.039	.034	.023	.026	.035	.043	.020	—.015	—.033	—.062	—.067	—.073	—.210	.00075
	.021	.017	.016	.014	.014	.012	.010	.008	.010	.015	.020	.024	.027	.084	
7	—.153	—.129	—.104	—.137	—.103	—.052	—.019	.072	.132	.152	.176	.117	.067	.524	.00000
	.125	.104	.098	.087	.082	.073	.062	.051	.062	.088	.122	.147	.164	.510	
8	—.052	—.048	—.034	—.016	.003	.031	.049	.027	.016	.009	.013	.014	—.004	—.041	.00045
	.018	.015	.014	.012	.012	.010	.009	.007	.009	.012	.017	.020	.023	.071	
								a_3							
0	—.118	—.123	—.124	—.111	—.101	—.090	—.057	—.011	.056	.113	.169	.206	.230	.524	.00025
	.010	.009	.008	.007	.007	.006	.005	.004	.005	.008	.011	.013	.015	.042	
1	.010	—.009	.007	.046	.034	.035	.033	.001	—.009	—.014	—.042	—.057	—.042	—.321	.00000
	.046	.040	.038	.035	.033	.030	.023	.017	.023	.035	.050	.060	.068	.213	
2	—.019	—.037	—.036	—.027	—.034	—.029	—.018	.004	.026	.038	.046	.049	.045	.258	.00000
	.025	.022	.021	.019	.018	.016	.013	.009	.012	.019	.028	.033	.037	.116	
3	—.208	—.159	—.143	—.107	—.101	—.079	—.046	—.008	.071	.133	.208	.237	.244	.967	.00000
	.038	.032	.030	.027	.026	.023	.019	.015	.018	.028	.039	.047	.053	.166	
4	—.119	—.122	—.128	—.113	—.092	—.086	—.067	—.022	.053	.117	.172	.208	.237	.590	.00011
	.021	.018	.017	.015	.014	.013	.010	.008	.010	.015	.022	.026	.029	.092	
5	—.134	—.133	—.139	—.132	—.119	—.118	—.075	—.023	.049	.121	.202	.258	.291	.850	.00031
	.020	.017	.016	.014	.013	.012	.010	.008	.010	.014	.020	.024	.027	.085	
6	—.082	—.093	—.102	—.113	—.115	—.112	—.076	—.042	.030	.097	.175	.227	.245	.672	—.00001
	.025	.021	.020	.018	.017	.016	.013	.010	.012	.019	.026	.031	.035	.110	
7	—.067	—.193	—.223	—.189	—.145	—.070	—.042	.010	.117	.153	.177	.233	.279	.332	.00000
	.098	.084	.081	.073	.069	.062	.049	.037	.048	.074	.105	.125	.141	.442	
8	.002	—.037	—.033	—.025	—.002	.002	.003	.026	.027	.022	.011	.006	.003	—.126	.00001
	.023	.020	.019	.017	.016	.015	.012	.009	.011	.017	.025	.029	.033	.104	

stems are generally thicker in northern that in southern Finland and are exceptionally thick on the wastelands; 'thickness' here and also later means the relative thickness after adjustment for size.

The coefficient of the second order term, $a_2(u)$, is in general positive for small angles and negative for large ones. Both small and large stems are thicker than stems of intermediate size (see Fig. 5). For regions 7 and 8 (the southwestern coast and the wastelands), however, the signs of $a_2(u)$ are opposite. This requires an explanation, even if the standard errors of the estimates are large

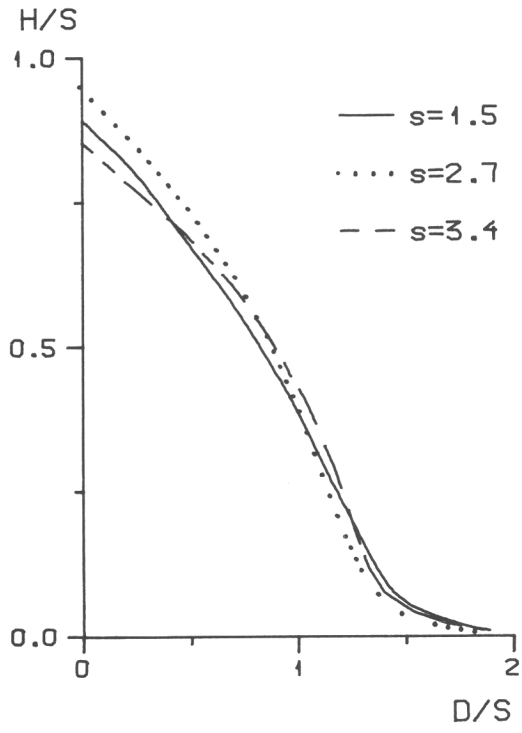

Fig. 5. Expected stem form according to the overall model for trees with sizes s = 1.5, 2.7 and 3.4; these sizes correspond to stem volumes 5.7 dm^3, 220 dm^3 and 1850 dm^3. The relative size is taken to be zero. To make the comparison of forms easier, dimensions are divided by the arithmetic size (S).

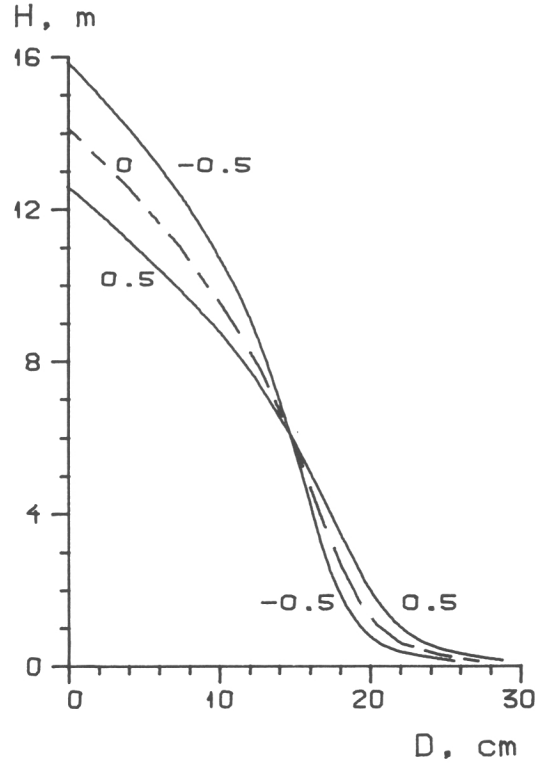

Fig. 6. Expected stem form for the size s=2.7 when the relative size is —0.5, 0. or 0.5; the overall model. Note that 0.5 is approximately twice the sample standard deviation of the relative size in the data.

compared with the estimates. The reason might be that in these regions the trees generally grow slowly and become thick. However, if the growing site is good, trees grow big and have a better stem form. Thus the controversial stem form may hold for the trees at a given time (i.e., cross-sectionally) but not for the stem form development of individual trees over their life span (i.e., longitudinally).

The parameter $a_3(u)$ is easier to interpret using equation (41.4) than with equation (41.3). In general, a_3 is negative for small angles and positive for large ones: stems that are bigger than others in the same stand are also thicker (see Fig. 6). For the two northern regions (regions 1 and 2), however, a_3 is not significantly different from zero. This indicates that the height competition is not so heavy in northern Finland as in southern Finland.

43. Covariance components

The variances and covariances of v's and e's for different angles, the covariance components, are estimated using the fitting constants method (Henderson's method 3). This method is described in general terms in Appendix A.3, and the computational details of our special case are given in Appendix B.1.

The estimated between-stand covariance matrix is denoted by **B** and the within-stand covariance matrix by **W**. The components of the matrices are written as:

$$B(u_1,u_2) = \mathrm{cov}[v_k(u_1), v_k(u_2)], \tag{43.1}$$

$$B(u) = \mathrm{var}[v_k(u)], \tag{43.2}$$

$$W(u_1,u_2) = \mathrm{cov}[e_{ki}(u_1), e_{ki}(u_2)], \text{ and} \tag{43.3}$$

$$W(u) = \mathrm{var}[e_{ki}(u)]. \tag{43.4}$$

Table 2. Estimated between-stand and within-stand standard deviations and correlations for the overall and regionalized models. The diagonal elements are standard deviations multiplied by 100 (corresponding to relative effects in percentages), the off-diagonal elements are correlations.

u	1	2	3	4	5	6	7	8	9	10	11	12	13	14
							Overall model							
							Between-stand							
1	7.11	.971	.929	.902	.889	.860	.754	.271	—.871	—.916	—.938	—.947	—.946	—.579
2		6.22	.980	.945	.910	.860	.708	.179	—.917	—.943	—.946	—.948	—.939	—.585
3			6.12	.989	.954	.909	.748	.200	—.942	—.974	—.967	—.960	—.945	—.658
4				5.71	.987	.956	.808	.275	—.938	—.988	—.979	—.967	—.955	—.726
5					5.57	.986	.876	.382	—.907	—.981	—.983	—.974	—.964	—.781
6						5.07	.940	.516	—.849	—.957	—.973	—.967	—.961	—.813
7							3.56	.775	—.632	—.831	—.886	—.890	—.891	—.782
8								1.84	—.006	—.316	—.419	—.444	—.460	—.481
9									3.14	.948	.893	.863	.839	.687
10										5.79	.986	.966	.948	.768
11											8.88	.995	.983	.748
12												10.76	.995	.709
13													12.42	.689
14														33.05
							Within-stand							
1	6.40	.806	.553	.419	.308	.177	—.024	—.284	—.593	—.654	—.641	—.606	—.573	—.280
2		5.18	.845	.636	.503	.343	.087	—.287	—.714	—.799	—.792	—.749	—.714	—.367
3			4.80	.806	.678	.491	.216	—.180	—.682	—.811	—.832	—.800	—.774	—.397
4				4.16	.830	.670	.388	—.031	—.603	—.788	—.840	—.831	—.806	—.421
5					3.93	.767	.498	.073	—.534	—.742	—.804	—.795	—.772	—.416
6						3.55	.596	.234	—.379	—.615	—.707	—.718	—.698	—.352
7							3.16	.546	—.028	—.356	—.501	—.545	—.557	—.269
8								2.75	.446	.140	—.097	—.190	—.228	—.045
9									3.20	.784	.587	.470	.410	.348
10										4.27	.864	.769	.714	.458
11											5.65	.938	.896	.480
12												6.77	.964	.427
13													7.53	.404
14														23.48
							Regionalized model							
							Between-stand							
1	5.35	.965	.900	.850	.824	.793	.699	.219	—.817	—.873	—.904	—.916	—.915	—.593
2		4.90	.971	.925	.887	.846	.703	.177	—.883	—.931	—.942	—.944	—.936	—.629
3			4.87	.990	.954	.923	.766	.206	—.916	—.975	—.972	—.962	—.948	—.715
4				4.55	.986	.966	.811	.257	—.907	—.981	—.975	—.959	—.949	—.760
5					4.36	.986	.864	.331	—.881	—.971	—.971	—.957	—.947	—.810
6						3.85	.929	.467	—.825	—.952	—.967	—.957	—.948	—.842
7							2.68	.761	—.583	—.823	—.887	—.890	—.881	—.797
8								1.44	.080	—.286	—.401	—.437	—.435	—.407
9									2.54	.931	.858	.815	.791	.721
10										4.61	.980	.951	.930	.809
11											6.95	.992	.976	.780
12												8.23	.994	.739
13													9.38	.713
14														29.61
							Within-stand							
1	6.29	.800	.538	.407	.294	.164	—.029	—.284	—.584	—.643	—.629	—.596	—.568	—.275
2		5.07	.838	.624	.490	.332	.082	—.289	—.710	—.792	—.783	—.741	—.709	—.366
3			4.70	.801	.671	.485	.214	—.179	—.678	—.804	—.826	—.795	—.771	—.398
4				4.10	.827	.666	.385	—.028	—.600	—.783	—.837	—.828	—.804	—.424
5					3.88	.765	.501	.083	—.527	—.737	—.803	—.795	—.772	—.424
6						3.53	.595	.236	—.379	—.613	—.705	—.716	—.694	—.359
7							3.15	.545	—.033	—.361	—.503	—.546	—.556	—.265
8								2.72	.437	.130	—.103	—.192	—.227	—.039
9									3.10	.778	.583	.469	.414	.365
10										4.18	.862	.767	.714	.467
11											5.53	.937	.895	.482
12												6.65	.964	.423
13													7.39	.402
14														23.11

Table 3. The first 6 characteristic vectors, latent roots (characteristic roots), percentages, and cumulative percentages of total variance absorbed by principal components for the overall and regionalized models.

	Overall model Components						Regionalized model Components					
u	1	2	3	4	5	6	1	2	3	4	5	6
	Between-stand						Between-stand					
1	.277	.313	—.542	—.516	.251	—.128	.263	.509	—.425	—.358	.402	—.086
2	.244	.419	—.175	.107	—.295	.235	.250	.428	—.073	.046	—.354	.143
3	.244	.309	.195	.320	—.286	.041	.254	.194	.244	.213	—.395	.057
4	.229	.119	.349	.225	.068	.009	.237	—.005	.345	.256	—.046	.049
5	.223	—.075	.332	.005	.339	—.336	.225	—.152	.315	.099	.368	—.375
6	.200	—.261	.262	—.117	.289	.047	.197	—.290	.205	—.040	.275	.064
7	.127	—.447	.020	—.267	—.099	.085	.124	—.404	—.094	—.335	.043	.069
8	.030	—.417	—.114	—.131	—.336	.320	.028	—.357	—.228	—.275	—.230	.344
9	—.115	—.273	—.265	.262	—.310	—.145	—.117	—.250	—.357	.134	—.334	—.142
10	—.232	—.105	—.326	.362	.124	—.492	—.239	—.052	—.346	.382	.029	—.494
11	—.362	.063	—.145	.266	.377	.242	—.368	.088	—.101	.379	.249	.303
12	—.439	.143	.111	—.087	.243	.495	—.437	.139	.187	—.003	.236	.447
13	—.504	.246	.337	—.435	—.361	—.369	—.495	.174	.375	—.501	—.244	—.378
Latent root×100	5.974	.147	.085	.022	.013	.003	3.518	.095	.073	.018	.010	.003
% of total variance	95.67	2.35	1.36	.35	.20	.05	94.64	2.56	1.96	.49	.26	.07
Cumulative %	95.67	98.02	99.39	99.74	99.94	99.98	94.64	97.20	99.16	99.65	99.92	99.99
	Within-stand						Within-stand					
1	.297	.622	—.450	—.352	—.204	.130	.294	.629	—.447	—.348	—.204	—.127
2	.291	.352	.052	.279	.331	—.320	.289	.356	.064	.274	.339	.316
3	.277	.087	.331	.500	.215	.011	.276	.085	.344	.495	.212	—.015
4	.237	—.114	.306	.078	—.337	.471	.238	—.115	.306	.070	—.346	—.465
5	.208	—.207	.278	—.200	—.353	.146	.210	—.213	.267	—.202	—.349	—.145
6	.160	—.273	.138	—.371	—.147	—.703	.162	—.276	.130	—.373	—.142	.703
7	.093	—.329	—.131	—.286	.499	.109	.095	—.329	—.144	—.278	.500	—.107
8	.004	—.301	—.317	—.033	.315	.242	.005	—.298	—.325	—.022	.309	—.250
9	—.134	—.208	—.379	.251	—.079	.065	—.133	—.198	—.374	.259	—.079	—.066
10	—.247	—.097	—.286	.341	—.306	—.157	—.247	—.091	—.280	.352	—.308	.167
11	—.361	.060	—.026	.146	—.213	—.158	—.361	.061	—.024	.142	—.209	.162
12	—.432	.181	.203	—.090	.073	.030	—.433	.176	.203	—.098	.075	—.032
13	—.469	.251	.331	—.268	.208	.135	—.471	.237	.331	—.275	.205	—.140
Latent root×100	2.295	.406	.174	.088	.053	.034	2.200	.398	.169	.086	.053	.034
% of total variance	72.50	12.82	5.50	2.78	1.68	.107	72.01	13.02	5.54	2.83	1.74	1.10
Cumulative %	72.50	85.32	92.82	93.60	95.28	96.34	72.01	85.02	92.56	93.39	95.12	96.23

This notation is used later in a generalized meaning when variances and covariances are interpolated for any angles.

Table 2 shows the between-stand and within-stand standard deviations and correlations for both the overall and the regionalized models. For the overall model the between-stand variances are larger than the within-stand variances. When using the regionalized model, the between-stand variances are reduced to approximately half. The within-stand variances are reduced only slightly, so that for the regionalized model between-stand variances are smaller than within-stand variances. Thus about half of the variation between stands is associated with the climatic regions.

The between-stand variance makes up a much greater part of the total variance for large angles than for small ones. It is also dominating in the crown height. This might be explained by the light competition which makes the heights and crown heights within a given stand more equal than the other dimensions. The stand effects are more correlated than the tree effects: the stand stem curves are more stable than the tree stem curves. This is quite natural, because the stand effects are just theoretical averages of the tree effects.

A better idea of a covariance structure is attained through the principal components (see, e.g., Morrison 1976). Thus the characteristic roots (= eigenvalues = latent roots)

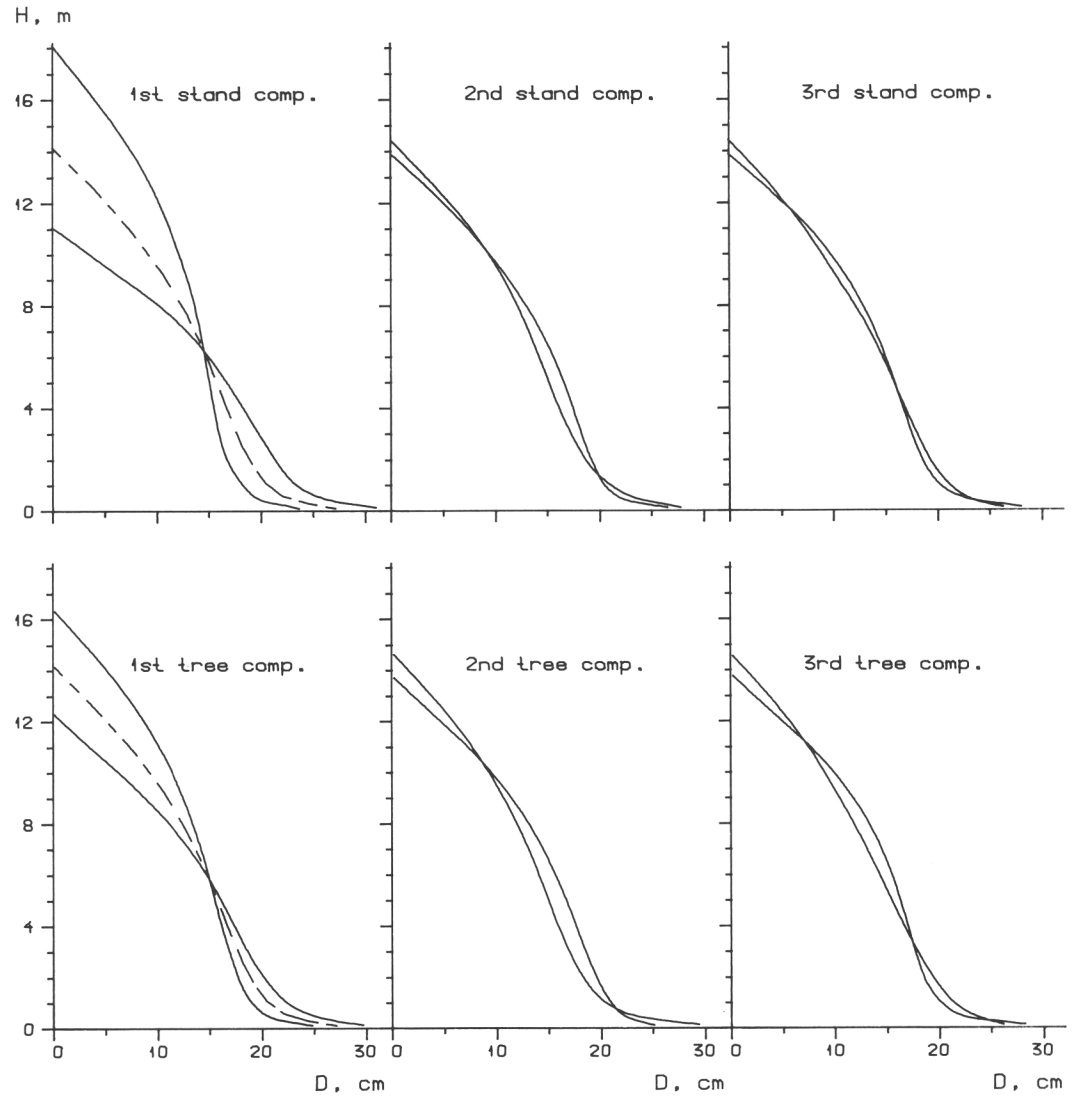

Fig. 7. Variation of stem forms in the directions of the first three characteristic vectors of the between-stand and within-stand covariance matrices. The overall model is used; s=2.7 and the relative size is zero. Each principal component is ± two times its standard deviation. In the subfigures for the first principal components, the expected stem curve is also shown (broken line).

and characteristic vectors (= eigenvectors = latent vectors) were computed from the estimated matrices for the between-stand and within-stand covariances. Note that principal components are merely used to illustrate the results of the analysis of the stem form variation; they are not 'the' analysis. Because we are interested in the variation of the stem form, the characteristic roots and vectors were computed from the covariances of pure stem dimensions, i.e., crown height was excluded. The first six characteristic vectors, latent roots (= variances of the principal components), percentages, and cumulative percentages of the total variance absorbed by each principal component are presented in Table 3 (total variance is the sum of the diagonal elements in a covariance matrix). The first three principal components of the covariance matrices of the overall model are illustrated in Fig. 7. Table 4 presents correlations of the principal components with the random effects (the crown height is included).

Table 4. Correlations of the principal components with the random effects for angles u=1,...,14 (u=14 is the crown height). The between-stand principal components are correlated with the random stand effects v(u), and the within-stand principal components are correlated with the random tree effects e(u).

	Overall model Components						Regionalized model Components					
u	1	2	3	4	5	6	1	2	3	4	5	6
	Between-stand						Between-stand					
1	.953	.169	—.222	—.107	.040	—.010	.924	.293	—.215	—.091	.074	—.008
2	.960	.258	—.082	.025	—.053	.020	.958	.270	—.040	.013	—.071	.015
3	.972	.193	.093	.077	—.052	.004	.978	.123	.135	.059	—.080	.006
4	.979	.080	.179	.058	.013	.001	.976	—.003	.205	.076	—.010	.006
5	.980	—.051	.174	.001	.068	—.033	.969	—.107	.195	.031	.083	—.044
6	.966	—.197	.151	—.034	.064	.005	.959	—.232	.144	—.014	.070	.009
7	.869	—.480	.016	—.111	—.031	.013	.865	—.465	—.095	—.169	.016	.013
8	.393	—.868	—.180	—.105	—.205	.094	.370	—.765	—.427	—.259	—.158	.122
9	—.894	—.334	—.246	.123	—.111	—.025	—.860	—.303	—.379	.971	—.129	—.028
10	—.978	—.069	—.164	.092	.024	—.046	—.971	—.035	—.202	.112	.006	—.055
11	—.996	.027	—.048	.044	.047	.015	—.995	.039	—.039	.074	.035	.022
12	—.998	.051	.030	—.012	.025	.025	—.996	.052	.061	—.001	.028	.028
13	—.992	.076	.079	—.052	—.033	—.016	—.989	.057	.108	—.072	—.026	—.021
14	—.700	.044	—.049	.018	—.002	—.003	—.389	.022	—.021	.011	—.004	—.001
	Within-stand						Within-stand					
1	.704	.619	—.293	—.163	—.073	.037	.693	.631	—.292	—.163	—.075	—.037
2	.851	.433	.042	.160	.147	—.114	.844	.443	.052	.159	.154	.114
3	.876	.116	.288	.310	.103	.004	.871	.115	.301	.309	.104	—.006
4	.863	—.175	.307	.056	—.187	.208	.860	—.177	.307	.050	—.194	—.208
5	.803	—.337	.295	—.151	—.207	.068	.802	—.346	.282	—.153	—.207	—.069
6	.684	—.490	.163	—.310	—.095	—.363	.682	—.494	.152	—.311	—.093	.366
7	.447	—.663	—.173	—.268	.365	.063	.450	—.660	—.188	—.260	.366	—.062
8	.023	—.698	—.482	—.035	.264	.162	.027	—.691	—.491	—.024	.261	—.168
9	—.636	—.414	—.495	.233	—.057	.037	—.634	—.402	—.495	.245	—.059	—.039
10	—.877	—.145	—.279	.237	—.165	—.068	—.875	—.138	—.276	.247	—.170	.074
11	—.968	.067	—.019	.077	—.087	—.051	—.967	.069	—.018	.076	—.087	.054
12	—.967	.170	.125	—.040	.025	.008	—.967	.167	.125	—.043	.026	—.009
13	—.944	.213	.184	—.106	.064	.033	—.945	.202	.184	—.109	.064	—.035
14	—.226	.009	—.013	.011	—.007	—.004	—.169	.007	—.011	.009	—.005	.003

The first 4—5 characteristic vectors are quite similar for both the between-stand and within-stand covariance matrices. The variation of the between-stand effects is more concentrated in the directions of the first few principal components. This can be seen already from the higher correlations in the between-stand covariance matrix. The first component is a 'thickness' component. When comparing Figs. 6 and 7, we see that the effect of relative size is very similar to the effect of the first principal component. The second component is associated with the variation in the middle of the stem.

Fries and Matérn (1966) also used principal components to analyze the stem form of birch. Their analysis is based on measurements made in arithmetic units. This explains their surprising result that the variation of butt swelling is the most important part in the variation of stem form. The principal components of Fries and Matérn and also of Liu and Keister (1978) were computed using the sample covariance matrix. The first component in their studies is the size component, hence almost all of the apparent variation is obviously in this direction. From this fact Liu and Keister incorrectly concluded that the other directions of the variation are not important.

Vector (33.1) used to define the size is one of the characteristic vectors of both the between-stand and within-stand covariance matrices up to the accuracy of three significant digits. For both matrices there is practically no variation (less than $4 \cdot 10^{-6}$ % of the total variance) in the direction of the size vector. This observation can be used to justify the approach to treat size as a fixed variable (a more detailed discussion will be gi-

ven in section 44).

Covariance matrices are always positive definite or at least positive semidefinite; a special case is the positiveness of variances. However, the obtained estimate of the between-stand covariance matrix **B** is positive definite neither for the overall nor for the regionalized model: the last three characteristic roots are negative, the absolute value of the smallest one being 0.03 % of the total variance. There is nothing intrinsic in the estimation method used to prevent the estimated matrix from being indefinite. These problems are widely discussed in the context of variance component estimation (see, e.g., Snedecor and Cochran 1980); Amemiya (1985) presents the problem in a form relevant to this case. See also Marquardt (1970) for a good discussion of the dimensionality problem in the context of usual regression analysis.

In the present case the negative characteristic roots are so small in absolute value that the problem of definiteness is more theoretical than practical. The between-stand covariance matrix is needed when the model parameters are estimated and, in a slightly different way, in the applications. In this study the definiteness problem is handled in three different ways:

i) When the fixed parameters are estimated separately for each angle only variances are needed and, therefore, the estimated indefinite matrix can be used as such.

ii) For small regions, fixed parameters are estimated simultaneously for all angles. Hence the covariance matrix must be inverted. In this case a small constant is added to the diagonal elements to obtain a positive definite matrix. This is essentially the 'ridge regression' solution to the singularity problem in the regression analysis (see, e.g., Marquardt 1970).

iii) In applications the problem is reformulated using the first few principal components, i.e., it is assumed that the variances of the other principal components are zero. This is analogous to replacing the negative variance estimates by zeros in the variance component estimation. The method of Amemiya (1985) for estimating covariance components is based on the same idea, but he defines the characteristic vectors and roots in the metric defined by the within-group mean square matrix. After this modification, or after adding a constant to the diagonal elements, the covariance matrix used is no longer an unbiased estimate of the true covariance matrix.

Let us briefly consider the distributions of the stand and tree effects; variances of the estimates of fixed parameters are ignored in the following discussion. When the fixed parameters are estimated, estimates of the random stand effects will also be obtained

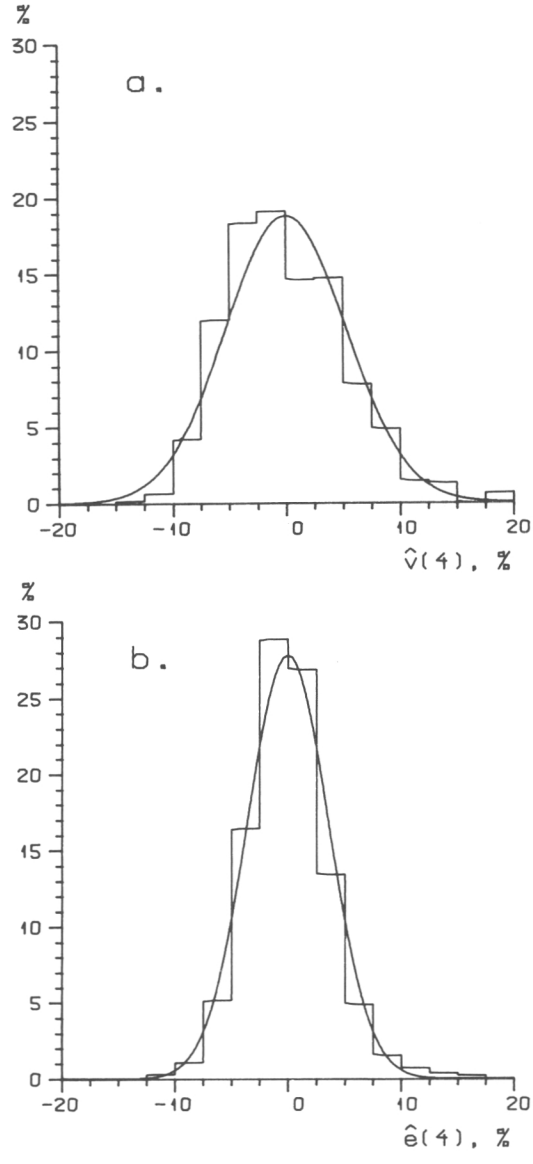

Fig. 8. Frequency histograms of the estimates of v(4) (a) and e(4) (b) and the corresponding normal density functions; the overall model.

(see Appendix A.2). Using the estimated parameters and stand effects, the tree effects can also be estimated (predicted) by the observed residuals. If the random stand and tree effects are normally distributed, their estimates are also normally distributed but do not have the same variances as the random effects themselves. Furthermore, the variances of the estimates vary from stand to stand depending on the number of trees in

the stand. The estimated random effects should, however, have a symmetric distribution. But their distributions are not exactly symmetric (Fig. 8): there are more very thick stands (stems) than very thin stands (stems). Therefore, neither the random stand effects nor the tree effects can follow precisely multinormal distributions, although deviations from normality are probably quite modest.

44. Consistency of the model

Let us return to the consistency of the model with respect to the definition of size. According to (41.3):

$$d_{ki}(u) = a_0(u) + a_1(u)s_{ki} + a_2(u)s_{ki}^2 + a_3(u)\bar{s}_{k.} + v_k(u) + e_{ki}(u), \qquad u=1,\ldots,13.$$

Multiply for each u=1,...,13 both sides of the equation by w(u) given in (33.2). If we add all these equations, the left side will be by definition (33.2) equal to s_{ki}. Thus we have:

$$s_{ki} = \sum_{u=1}^{13} w(u)d_{ki}(u) = \sum_{u=1}^{13} w(u)a_0(u) + \sum_{u=1}^{13} w(u)a_1(u)s_{ki} + \sum_{u=1}^{13} w(u)a_2(u)s_{ki}^2 + \sum_{u=1}^{13} w(u)a_3(u)\bar{s}_{k.} + \sum_{u=1}^{13} w(u)v_k(u) + \sum_{u=1}^{13} w(u)e_{ki}(u). \qquad (44.1)$$

The system is theoretically consistent if the above equation is always true, i.e., identically true with respect to the fixed variables (s^2, s, $\bar{s}$ and the constant term), and true with probability one with respect to random variables (v and e).

In section 43 vector $\mathbf{w} = [w(1),\ldots w(13)]'$ was found to be among the characteristic vectors (up to a multiplicative constant) of both between-stand and within-stand effects, and the associated characteristic root indicated practically no variation in this direction. Thus the following consistency requirements seem to be fulfilled rather closely:

$$\sum_{u=1}^{13} w(u)v_k(u) = 0, \text{ and} \qquad (44.2)$$

$$\sum_{u=1}^{13} w(u)e_{ki}(u) = 0. \qquad (44.3)$$

Equation (44.1) is identically true with respect to the fixed variables, if the fixed parameters satisfy the constraints:

$$\sum_{u=1}^{13} w(u)a_0(u) = 0, \qquad (44.4)$$

$$\sum_{u=1}^{13} w(u)a_1(u) = 1, \qquad (44.5)$$

$$\sum_{u=1}^{13} w(u)a_2(u) = 0, \text{ and} \qquad (44.6)$$

$$\sum_{u=1}^{13} w(u)a_3(u) = 0. \qquad (44.7)$$

These sums were computed using the estimated parameter values for the overall model and for different regions in the regionalized model; the results were given in Table 1. These sums are sufficiently close to the required values to justify this approach. The constraints should be formally tested using the multivariate model. For computational reasons, however, the multivariate model can be used only for some regions in the regionalized model, for which the constraints are so closely satisfied that they need no testing. For the overall model or for large regions in the regionalized model, comparison with the standard deviations of the parameter estimates indicates strongly that the validity of the constraints would also pass a formal test. It was not considered to be necessary to re-estimate the parameters subject to the constraints (44.4)—(44.7).

5. APPLICATION TECHNIQUES

51. Reversing the role of variables and parameters

After the analysis stage, we can assume that we know the variances and covariances of the stand effects (v) and the tree effects (e), as well as the values of the fixed parameters (a_0, a_1, a_2, a_3) at the knot angles. It is assumed further that we can interpolate the fixed parameters and covariances for all other angles by one- or two-dimensional cubic splines. In applications of the model, stem curves are to be predicted if any dimensions are measured from trees in a given stand. This chapter describes how standard linear methods can be used to predict stem curves and stem volumes.

Using the second formulation (41.4) of the model, a measured diameter $d(u_{ij})$ of tree i and measurement angle u_{ij} can be expressed:

$$d(u_{ij}) = a_0(u_{ij}) + a_1^*(u_{ij})s_i + a_2(u_{ij})s_i^2 - a_3(u_{ij})(s_i - \bar{s}) + v(u_{ij}) + e(u_{ij}). \quad (51.1)$$

Stand index k is dropped because we are considering a given stand. The unknowns are s_i (for each tree i), $\bar{s}$ (for the stand) and $v(u_{ij})$ (for each measurement angle in the stand). During the analysis stage, s_i and $\bar{s}$ were known and the a-parameters were estimated. In applications we assume that the a-parameters are known, and s_i and $\bar{s}$ are being estimated. Thus the role of parameters and variables is changed. The measured dimensions enter into the model as dependent variables and not as explanatory variables, as in the standard regression approach.

The equation is linear with respect to the unknowns s_i^2, s_i and $\bar{s}$. However, there exists a nonlinear constraint for the unknowns s_i^2 and s_i, i.e., $s_i^2 = s_i s_i$. But the $a_2(u)$-coefficients are so small that the function $a_2(u)s^2 + a_1(u)s$ is almost linear and can be approximated very accurately by the first order Taylor series. The approximation is the better the closer s_i is to the point of expansion, denote it $\dot{s}_i$. A rather good estimate of s_i is obtained by taking the random effects (v and e) and the relative size to be zero in (51.1) and by solving the resulting quadratic equation for s_i. If several dimensions have been measured for tree i, then the preliminary estimate $\dot{s}_i$ is computed from the lowest measured diameter (the average could also be used).

The first order Taylor series estimate for $a_2(u_{ij})s_i^2$ is then:

$$a_2(u_{ij})s_i^2 \simeq -a_2(u_{ij})\dot{s}_i^2 + 2a_2(u_{ij})\dot{s}_i s_i. \quad (51.2)$$

The preliminary estimates $\dot{s}_i$ are also used to estimate average size ($\bar{s}$) and relative sizes ($s_i - \bar{s}$). When these approximations are made, the model (51.1) can be written in the form:

$$d(u_{ij}) - a_0(u_{ij}) + a_2(u_{ij})\dot{s}_i^2 + a_3(u_{ij})(\dot{s}_i - \bar{s}) = [2a_2(u_{ij})\dot{s}_i + a_1^*(u_{ij})]s_i + v(u_{ij}) + e(u_{ij}). \quad (51.3)$$

Denote the left side by $y(u_{ij})$ and the coefficient of s_i by $a(u_{ij})$, i.e.,

$$y(u_{ij}) = d(u_{ij}) - a_0(u_{ij}) + a_2(u_{ij})\dot{s}_i^2 + a_3(u_{ij})(\dot{s}_i - \bar{s}), \text{ and} \quad (51.4)$$

$$a(u_{ij}) = 2a_2(u_{ij})\dot{s}_i + a_1^*(u_{ij}). \quad (51.5)$$

Then (51.3) can be expressed as an ordinary mixed linear model

$$y(u_{ij}) = a(u_{ij})s_i + v(u_{ij}) + e(u_{ij}). \quad (51.6)$$

The model contains ordinary random stand effects v(u). A better way to take the random stand effects into account will be discussed in the next section.

52. Describing random stand effects using principal components

Let us then consider the estimation of the random v-parameters. If diameters have been measured at the same angles in differ-

ent stems, we could just use the standard estimation method described in Appendix A.2. But in practice only the height and crown height will be measured at 'fixed' angles. Other diameters are usually measured at absolute heights, which correspond to different angles in the polar coordinate system. Although it would be technically possible to estimate a separate v-term for each measurement angle, this would not be reasonable computationally or for the construction of the final stem curves.

As noted, the dimensionality of the stand effects is low. Already 3 (5) first principal components absorbed over 99 % (99.9 %) of the total between-stand variance both in the overall and regionalized models (Table 3). Thus v-effects can be expressed virtually without error in terms of 3—5 first (unknown) principal components.

Let us first consider the determination of the v-effects at the knot angles. Let $\mathbf{v}' = [v(1),\ldots, v(13)]$ be the vector of the v-effects, let $\mathbf{q}_k$ be the k^{th} characteristic vector with elements $q_k(u)$, $u=1,\ldots,13$, and let $\mathbf{Q}$ be the matrix having the characteristic vectors as its rows. Denote the vector of the principal components by $\mathbf{c}$ where $c_k = \mathbf{q}_k'\mathbf{v}$. Then

$$\mathbf{c} = \mathbf{Q}\mathbf{v}. \tag{52.1}$$

Since $\mathbf{Q}$ is orthogonal,

$$\mathbf{v} = \mathbf{Q}'\mathbf{c}. \tag{52.2}$$

If only the first p characteristic roots are nonzero, then $\text{var}(c_k) = 0$ for $k = p+1,\ldots,13$, and hence $c_k = 0$, $k = p+1,\ldots,13$ with probability 1. Thus for $u = 1,\ldots,13$

$$v(u) = \sum_{k=1}^{p} q_k(u)c_k. \tag{52.3}$$

When using the estimated characteristic vectors, the above expression is only approximative.

The same expression (52.3) is obtained when v(u) is regressed on the first p principal components. This can be seen by noting that the principal components are uncorrelated, $\text{var}(c_k) = t_k$ (t_k being the k^{th} characteristic root), and $\text{cov}[v(u),c_k] = t_k q_k(u)$. Thus, when regressing v(u) on c_k the regression coefficient is $\text{cov}[v(u),c_k]/\text{var}(c_k) = q_k(u)$.

We should, however, express a v-effect at any angle in terms of principal components. The v-effects change smoothly with the angle u, so we may assume that the coefficients $q_k(u)$ can be interpolated by cubic splines. The first 5 vectors seem to be smooth enough for interpolation. Denote by $q_k(u_{ij})$ the (interpolated) coefficient of the k^{th} principal component at angle u_{ij}. Then:

$$v(u_{ij}) = \sum_{k=1}^{p} q_k(u_{ij})c_k. \tag{52.4}$$

The model can now be stated for the j^{th} measurement of tree i:

$$y(u_{ij}) = a(u_{ij})s_i + \sum_{k=1}^{p} q_k(u_{ij})c_k + e(u_{ij}). \tag{52.5}$$

Finally the model is in a form where standard estimation procedure for mixed linear models can be applied. The parameters to be estimated are the size s_i for each tree i and the values of the first p principal components, i.e., c_k, $k=1,\ldots,p$. Because the c_k-parameters are random, c_k- and s_i-parameters can also be estimated in case there are fewer measurements than parameters. The special matrix structures needed in the parameter estimation are presented in Appendix C.1, including the case where the sizes of stems are assumed to be random parameters. If the size is considered to be random, the model does not contain any fixed parameters.

At an earlier stage of the study, $v(u_{ij})$'s were expressed in terms of $v(1),\ldots, v(13)$, i.e., the v-effects at the knot angles were the random effects to be estimated as in the analysis stage. Results were virtually the same. The approach based on the principal components was chosen for computational reasons: smaller linear systems need to be solved.

We will next consider prediction of diameters at the knot angles. When deriving prediction formulas, the size is assumed to be a fixed parameter. The formulas can easily be extended to the case of random size. The characterization of size as a random parameter is discussed later in section 71.

53. Stem curve and volume of an individual tree

At this point we assume that we have estimated the size s_i of stem i and the first p principal components of the stand effects, c_k, $k=1,\ldots,p$. The problem is then to predict the stem curve of stem i at knot angles

u=1,...,13. The principal task is to predict the y-variable defined in (51.4) for the knot angles. The predictor and its error variance are derived in detail in Appendix C.2. The predictor of the y-variable for stem i at angle u is found to be:

$$\hat{y}_i(u) = a(u)\hat{s}_i + z(u)'\hat{c} + w_i'R_i^{-1}(y_i - \hat{s}_i a_i - Z_i\hat{c}),$$

where $z(u)' = [q_1(u),\ldots, q_p(u)]$ tells how the random stand effect v(u) is obtained from the principal components; $\hat{c}' = (\hat{c}_1,\ldots,\hat{c}_p)$ is the vector of estimated principal components of the stand effects; $w_i' = [W(u,u_{i1}),\ldots, W(u,u_{i,m_i})]$ contains the covariances between $e_i(u)$ and the e-terms of the measured dimensions; $R_i = \mathrm{var}[e(u_{i1}),\ldots, e(u_{i,m_i})]$ is the variance matrix of the e-terms of the measured dimensions, and $(y_i - \hat{s}_i a_i - Z_i\hat{c})$ is the vector of residuals for stem i, when the measured dimensions are predicted by the stand stem curve, i.e., using the estimated random stand effects (and size). The predicted log-diameter is obtained from $\hat{y}_i(u)$ using definition (51.4):

$$\hat{d}_i(u) = \hat{y}_i(u) + a_0(u) - a_2(u)\bar{s}_i^2 - a_3(u)(\bar{s}_i - \bar{s}) \qquad (53.1)$$

Omitting the complications caused by the $\bar{s}$- and s_i^2-terms in the model, the prediction of diameters at the knot angles is easy to understand. We calculate the deviations of the measurements from the predicted values obtained using the estimated size of the stem and the estimated random stand effects. Deviations at the knot angles are then predicted for a given stem using the observed deviations as explanatory variables; the regression coefficients are obtained from the within-stand covariance matrix. Thus the observed residuals are used to predict the 'unobserved residuals' in the same way as known e-terms at the measurement angles would be used to predict other e-terms.

If size is a fixed parameter, then the predictors of the logarithmic diameters are unbiased for the given (unknown) size. If, in addition to stand and tree effects, also size is normally distributed with a known mean and variance, then the predictors are conditionally unbiased for the measurements. It is not possible to have a predictor that is conditionally unbiased both for the given unknown size and the measurements.

Determination of unbiased predictors in the arithmetic scale is based on the assumptions of normality. Let y be a normally distributed random variable with mean μ and variance σ^2, then $z = e^y$ is a lognormally distributed variable having t^{th} moment (e.g., Flewelling and Pienaar 1981):

$$E(z^t) = \exp(t\mu + \frac{1}{2}\, t^2\sigma^2). \qquad (53.2)$$

From this formula we get the mean of z (t = 1):

$$E(z) = \exp(\mu + \tfrac{1}{2}\sigma^2). \qquad (53.3)$$

If size s is fixed, we will get unbiased predictors of diameters in the logarithmic scale for the given unknown size. In this case μ is the fixed part of the model (41.3). To simplify the following expressions, angle u is not explicitly written in D(u), d(u), y(u), v(u), or e(u). Now,

$$E(d) = E(\hat{d}) = \mu,$$

$$E(D) = E[\exp(d)] = \exp[\mu + \tfrac{1}{2}\mathrm{var}(d)], \text{ and}$$

$$E[\exp(\hat{d})] = \exp[\mu + \tfrac{1}{2}\,\mathrm{var}(\hat{d})]$$

Note that $\mathrm{var}(d) = \mathrm{var}(v) + \mathrm{var}(e) = B(u) + W(u)$ and $\mathrm{var}(\hat{d}) = \mathrm{var}(\hat{y})$. As var(d) is not generally equal to var($\hat{d}$), $\exp(\hat{d})$ is biased for D. An unbiased predictor for D is obtained by:

$$\hat{D} = \exp\{\hat{d} + \tfrac{1}{2}[\mathrm{var}(d) - \mathrm{var}(\hat{d})]\}. \qquad (53.4)$$

For the normally distributed random size we can get predictors of diameters in the arithmetic scale that are conditionally unbiased for the fixed measurements. In this case $\hat{d}$ is the conditional mean and var($\hat{d}$—d) the conditional variance. Thus a conditionally unbiased predictor in the arithmetic scale is obtained by:

$$\hat{D} = \exp[\hat{d} + \tfrac{1}{2}\,\mathrm{var}(\hat{d} - d)] \qquad (53.5)$$

Using formula (53.4) (or 53.5, if applicable) we get unbiased predictions for the diameter at a given angle. We may also be interested in getting an unbiased predictor for the diameter at a given height. If we convert the predictor $\hat{D}(u)$ directly from the polar coordinates to the corresponding predictor $\hat{D}_H(H)$ expressed as a function of height, then $\hat{D}_H(H)$ is biased for a given height un-

less the stem curve is linear. Computational experiments with the predicted stem curves indicated, however, that this bias is negligible.

One of the main applications of the stem curve model is to predict the volume of the whole stem or different stem segments. The first problem in determining volumes is terminological: Should we 'predict' or 'estimate' volumes? We 'predict' diameters at different angles, and the volume is determined by integrating squared diameters, i.e., 'predicted' cross-sections. Thus we are also 'predicting' volumes. But the stem volume is very closely associated with the size of the stem, and size is (for most of the time) treated as a fixed parameter. Hence we could also 'estimate' the volume. In the following, volume is 'predicted', if the integration of the 'predicted' cross-sections is emphasized. Otherwise volume is 'estimated'.

Let us then consider how to obtain an unbiased volume predictor from the predicted logarithmic diameters. The volume is predicted by integrating the cross-sectional areas. In case the stem size is assumed to be fixed, the volume predictor is unbiased if:

$$E(\widehat{D^2}) = E(D^2).$$

From (53.2) we get

$$E(D^2) = \exp(2\mu + 2\sigma^2).$$

Using arguments similar to those above for the unbiased predictor of D, an unbiased predictor for D^2 is found to be:

$$\widehat{(D^2)} = \exp\{2\hat{d} + 2[\mathrm{var}(d) - \mathrm{var}(\hat{d})]\}.$$

Thus the corresponding 'volume unbiased' predictor of D is:

$$\hat{D} = \exp[\hat{d} + \mathrm{var}(d) - \mathrm{var}(\hat{d})]. \qquad (53.6)$$

Similarly, for random size the 'volume unbiased' predictor is:

$$\hat{D} = \exp[\hat{d} + \mathrm{var}(\hat{d} - d)]. \qquad (53.7)$$

As the polar coordinate system was defined by expressing height in meters and diameter in centimeters, the volume cannot be obtained by integrating the stem curve directly over the angle. Any function defined in polar coordinates can, however, be converted to the corresponding function, where height is the argument variable. Then the integration can be done in the usual height-diameter coordinate system. Integration over height can also be applied when predicting volumes of different stem segments and is easy to compute. With splines the conversion from polar coordinates to height coordinates is simple. First the predictor $\hat{D}(u)$ is computed for the knot angles. Then points $(\hat{D}(u)\tan(u), \hat{D}(u))$ are used as the knot points when defining the stem curve splines in height coordinates.

When the predicted stem curve of a thick stem is converted from polar coordinates to height coordinates, the heights corresponding to adjacent angles can be very near to each other. If cubic splines are then defined expressing the knot points in height coordinates, these splines can have wild oscillations. To prevent this oscillation, points were merged whenever they were closer than half a percent of the total height of the tree. The arithmetic mean of the height and diameter coordinates seemed to work as well as the more sophisticated methods tested.

The estimation procedure can be applied if any combination of dimensions is measured for different trees in the stand. However, the computations can be simplified if only one dimension is measured for some trees and size is treated as fixed. According to forestry practice, trees with one measured dimension are called tally trees. A single measured dimension is needed to estimate the size; it does not contain any information about the stem form. The estimation can thus be made in two different stages. First, we ignore trees with one measured dimension and estimate the stand effects, sizes and stem forms as described earlier. In the second stage we estimate the sizes and stem forms of trees with one measured dimension using the estimated stand stem curve (estimated random stand effects). This intuitively sound procedure is presented more formally in Appendix C.3.

In many stem curve problems error variances are also needed. For the predicted stem curve, variance of the relative errors is given by $\mathrm{var}(d - \hat{d})$. The exact derivation of the volume estimation errors would be rather difficult, because the volume is predicted by integrating the stem curve in arithmetic units. An approximative description is based on the close relation between the size pa-

rameter and the volume of the stem.

The total variance and the variance components of $\hat{s}$ are easily obtained. The total variance of $\hat{s}_i$ is just the i^{th} diagonal element of the inverse of the coefficient matrix **H** of the linear system formed to estimate the size parameters and the stand effects (see Appendix A.2 and C.1). As discussed in general terms in Appendix A.2, any $\hat{s}_i$ is a linear combination of the observed y's, and hence also of the within-stand random effects e of all trees and measurements in the stand. Then, the within-stand variance of $\hat{s}_i$ is defined as the variance of $\hat{s}_i$ with respect to the within-stand random effects $e(u_{ij})$ of the same tree i: the within-stand variance is the conditional variance of $\hat{s}$ given the random stand effects and the random within-stand effects of other trees. The between-stand variance is then obtained by subtracting the within-stand variance from the total variance.

More precisely, $\hat{s}_i$ is of the form

$$\hat{s}_i = \mathbf{t}_i'\mathbf{e}_i + \text{additional terms},$$

where $\mathbf{t}_i$ is some vector and

$$\mathbf{e}_i' = [e(u_{i1}), \ldots, e(u_{i,m_i})].$$

Thus the variance of $\hat{s}_i$ with respect to $\mathbf{e}_i$, $\text{var}_w(\hat{s}_i)$, is

$$\text{var}_w(\hat{s}_i) = \mathbf{t}_i'\mathbf{R}_i\mathbf{t}_i, \tag{53.8}$$

where $\mathbf{R}_i = \text{var}(\mathbf{e}_i)$.

The derivation of an explicit expression for $\mathbf{t}_i$ in terms of the quantities of Appendix C.1 is straightforward but less informative and is omitted here. For the tally trees (i.e., trees with one measured dimension) the variance components can be derived more directly. If only one dimension has been measured for tree i, i.e., the diameter at angle u_{i1}, then according to (C.3.10) $\hat{s}_i$ is:

$$\hat{s}_i = e(u_{i1})/a(u_{i1}) + \text{other terms},$$

where $a(u_{i1})$ is defined according to (51.5). Thus in this case the within-stand variance is

$$\text{var}_w(\hat{s}_i) = W(u_{i1})/a(u_{i1})^2, \tag{53.9}$$

where $W(u_{i1}) = \text{var}[e(u_{i1})]$. The total variance of $\hat{s}_i$ for this special case is given in (C.3.12).

After computing the within-stand variance $\text{var}_w(\hat{s}_i)$ and the total variance $\text{var}_t(\hat{s}_i)$, the between-stand variance $\text{var}_b(\hat{s}_i)$ is simply:

$$\text{var}_b(\hat{s}_i) = \text{var}_t(\hat{s}_i) - \text{var}_w(\hat{s}_i). \tag{53.10}$$

Because the within-stand errors are uncorrelated for different trees, we have:

$$\text{cov}_w(\hat{s}_{i_1}, \hat{s}_{i_2}) = 0, \text{ and} \tag{53.11}$$

$$\text{cov}_b(\hat{s}_{i_1}, \hat{s}_{i_2}) = \text{cov}_t(\hat{s}_{i_1}, \hat{s}_{i_2}) \text{ for } i_1 \neq i_2. \tag{53.12}$$

If the measurements are made at absolute heights, we get different estimates for between-stand and within-stand variances for different trees. Thus the error of $\hat{s}_{ki}$ is of the form

$$s_{ki} - \hat{s}_{ki} = b_{ki} + e_{ki}.$$

The theoretical estimates of variance components are later compared with empirical estimates. In the empirical descriptive model we have a single between-stand error for all trees in the same stand. The theoretical between-stand variance is more comparable with the empirical one if we write first:

$$s_{ki} - \hat{s}_{ki} = \bar{b}_{k.} + (b_{ki} - \bar{b}_{k.} + e_{ki}). \tag{53.13}$$

Thereafter, the between-stand variance can be interpreted as:

$$\text{var}_b(s_{ki} - \hat{s}_{ki}) = \text{var}(\bar{b}_{k.}) = \tag{53.14}$$

$$[\sum_{i_1}\sum_{i_2}\text{cov}[b_{ki_1}, b_{ki_2})]/n^2.$$

In this way the between-stand variance will be the same for all trees in the stand.

The variance components of the logarithmic volume errors (or relative errors of volume estimates) can be obtained from the respective variance components of the size parameters using the relation $\ln(V) \simeq 3.042s$ + constant (Eq. 33.4):

$$\text{var}[\ln(V) - \ln(\hat{V})] \simeq \text{var}[(V - \hat{V})/E(V)] \simeq 3.042^2\text{var}(\hat{s}). \tag{53.15}$$

54. Measurement errors

In practice, all measurements contain errors. If the true diameter is defined to be the diameter of the circle having correct cross-sectional area, then the noncircular form brings about errors that are comparable to 'pure' measurement errors (see Matérn 1956). Let $d_m(u)$ be the measured diameter in logarithms for the angle u, then:

$$d_m(u) = d(u) + e_m(u), \tag{54.1}$$

where $e_m(u)$ is the measurement error in the polar coordinates. A later discussion will indicate how $e_m(u)$ relates to the corresponding measurement error in the height-diameter coordinates.

Let us assume that

$$E[e_m(u)] = 0, \text{ and} \tag{54.2}$$

$$\text{var}[e_m(u)] = \sigma_m^2(u). \tag{54.3}$$

Furthermore, measurement errors are assumed to be mutually uncorrelated and also uncorrelated with the random stand or tree effects. The variance of the measurement errors may be a function of d(u). For practical purposes, the variance may be equally well determined as a function of $d_m(u)$. For instance, if the error variance is a constant c in the arithmetic scale for a diamater measurement D_m, then, using the first order Taylor series approximation, the logarithmic error variance is

$$\sigma_m^2 \simeq c/D_m^2. \tag{54.4}$$

If measurements are unbiased in the arithmetic scale, they have a bias in the logarithmic scale; this bias is, however, negligible for realistic measurement errors (see section 53 for the connection between the expectations in the arithmetic and logarithmic scales).

When the stem sizes and random stand effects are estimated, the random tree effect e(u) and the measurement error $e_m(u)$ together form a combined random tree effect, denoted by $e_*(u)$:

$$e_*(u) = e(u) + e_m(u). \tag{54.5}$$

This new tree effect behaves exactly like the previous e(u). The only difference is that the variance of the measurement errors will increase the variance:

$$\text{var}[e_*(u)] = \text{var}[e(u)] + \text{var}[e_m(u)] = W(u) + \sigma_m^2(u). \tag{54.6}$$

The covariances remain unchanged:

$$\text{cov}[e_*(u_1), e_*(u_2)] = \text{cov}[e(u_1), e(u_2)] = W(u_1,u_2). \tag{54.7}$$

Thus the new within-stand covariance function has a jump on the line $u_1=u_2$. In short, the stem sizes and the random stand effects are estimated as described in Appendix C, the only difference being that we add the variances of the measurement errors to the diagonal elements of **R**, the covariance matrix of the within-stand effects.

The stem curves are then predicted as previously using the formula (C.2.8). The diagonal elements of $\mathbf{R}_i$ are only augmented by the variances of the measurement errors, as in the estimation of the stem sizes and stand effects. Because the measurement er-

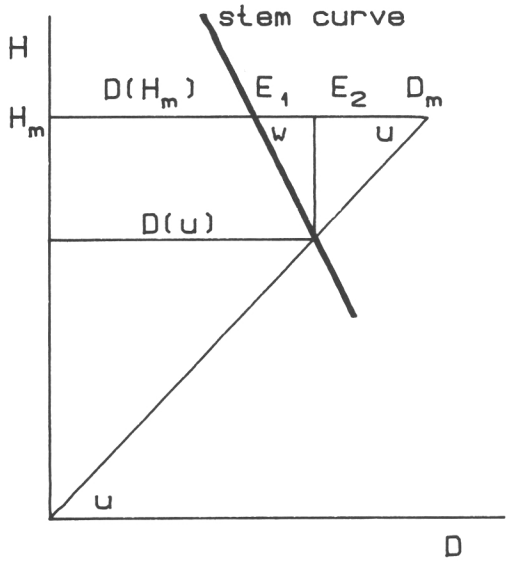

Fig. 9. Measurement errors in the polar coordinates differ from those in the height-diameter coordinates. H_m is the height of the measurement, D_m is the diameter at H_m according to the erroneous measurement, $D(H_m)$ is the true diameter at H_m, u is the angle of the measurement in the polar coordinates, and D(u) is the true diameter at angle u. The measurement errors are E_1+E_2 and E_2 in the height-diameter coordinates and in the polar coordinates, respectively.

rors do not change the within-stand covariances, the vector $\mathbf{w}_i$ is the same as when no measurement errors occur.

In practice, variances of the measurement errors are evaluated in the height-diameter coordinates. The measurement errors for height are the same in both coordinate systems. Measurement errors of diameters in the height-diameter coordinates can be transformed to the polar coordinate system using the following approximative procedure. Assume that the stem curve is linear around the height of the measured diameter. Denote according to Fig. 9: w is the angle between the stem curve and the horizontal axis, u is the angle of the measured point in the polar coordinates, $E=E_1+E_2$ is the measurement error in the height-diameter coordinates, and E_2 is the measurement error in the polar coordinates for angle u. Then:

$E_1\tan(w) = E_2\tan(u)$, or

$E_1 = E_2\tan(u)/\tan(w)$.

Hence

$E = E_1+E_2 = [1+\tan(u)/\tan(w)]E_2$, or

$E_2 = fE$, where (54.8)

$f = [1+\tan(u)/\tan(w)]^{-1}$ (54.9)

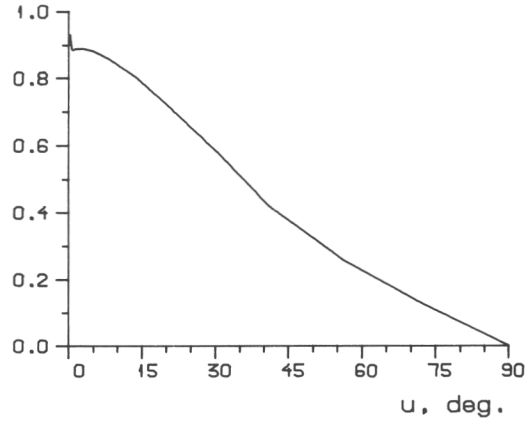

Fig. 10. Coefficient f defined in (54.9) as a function of u.

Thus the standard deviation of the measurement errors in the polar coordinates is the usual standard deviation multiplied by f defined in (54.9), where tan(w) can be computed from the average stem curve. Fig. 10 shows f as a function of u.

In the presence of measurement errors, the predicted stem curves do not go through the measured points (Fig. 11): the measurement errors can be corrected to some extent. In addition, the stem form model can be used to analyze the effect of measurement errors when different measurement devices and strategies are compared.

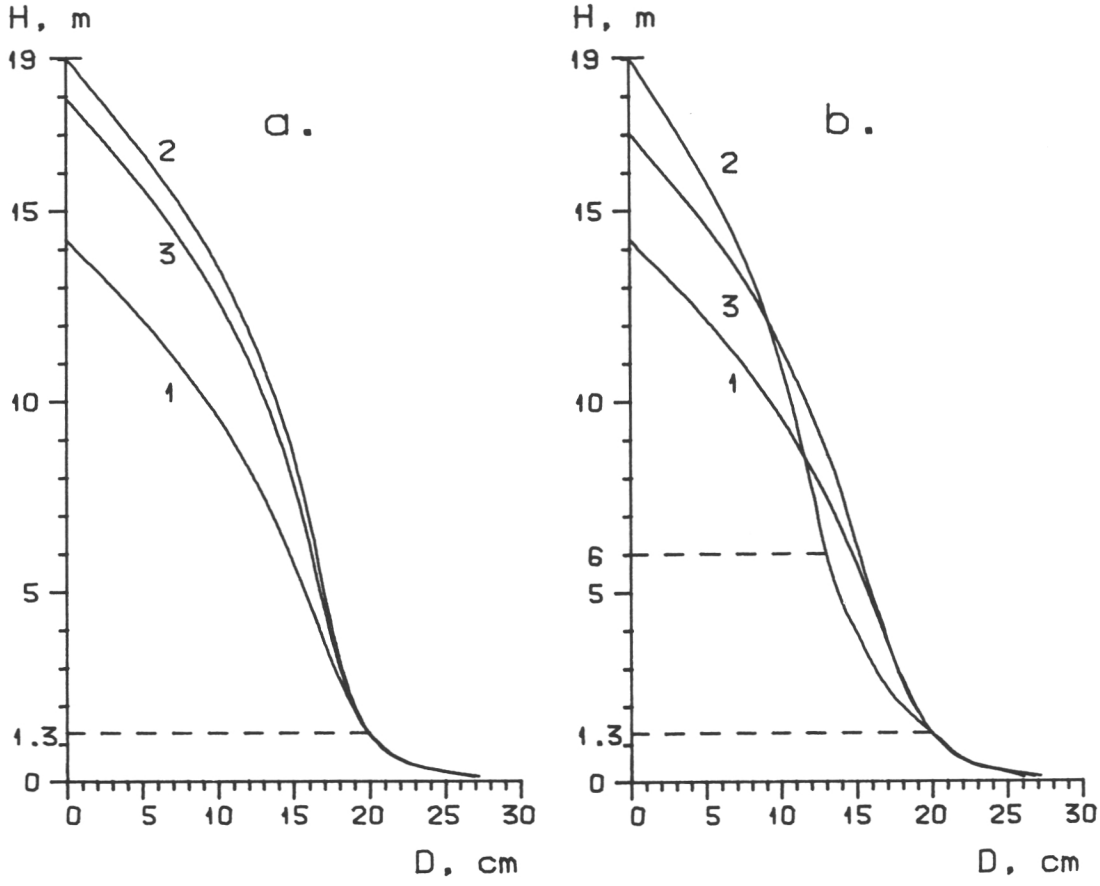

Fig. 11. When measurements contain errors, the predicted stem curve is shifted towards the average stem curve. Assume that $D_{1.3}$ is measured without error. In Fig. 11a the predicted stem curves are for measurements: (1): $D_{1.3}$=20 cm; (2): $D_{1.3}$=20 cm, H=19 m, and H is measured without error; (3): $D_{1.3}$=20 cm, H=19 m, and the standard error of measuring H is 10 %. In Fig. 11b the predicted stem curves are: (1): $D_{1.3}$=20 cm; (2): $D_{1.3}$=20 cm, H=19 m, D_6=13 cm, and the measurements contain no error; (3): measurements are as in (2), but the standard errors of measuring H or D_6 are both 10 %.

6. TEST RESULTS

61. Test criteria

The usefulness of a stem curve model is evaluated by its performance in practical applications. This chapter describes results of tests where the stem form model is used to predict stem curves and volumes, the standwise calibration of the stem curve (section 65) having a special emphasis. The stem form model is used in its basic form, i.e., the sizes are taken to be fixed parameters, and the measurements contain no errors. Random size parameter, measurement errors, and also application of estimated variances in timber assortment problems and in the optimization of measurements are considered in the next chapter. Let us first discuss how the performance of a stem curve model can be evaluated.

The measurement scale causes problems when a stem curve model is applied. The prediction of the stem curve should work properly in the arithmetic scale, e.g., the predictors should be unbiased in the arithmetic scale (see section 53). The prediction variance in the arithmetic scale is, however, closely associated with the size of the trees.

Error variances and observed biases give a better idea of the overall performance, if they are computed in the logarithmic scale. If, however, the predictions are in fact computed in the arithmetic scale and the errors are compared in the logarithmic scale without the bias correction for the arithmetic scale, there will result a slight gap between the predictor and the evaluation criterion. One possibility to overcome the problem is to use relative errors:

$$e_r = (y - \hat{y})/E(y), \tag{61.1}$$

where y is the true value, $\hat{y}$ is the predicted value, and E(y) is the expected value of y.

The relative error e_r is the first order Taylor approximation of the logarithmic error $\ln(y) - \ln(\hat{y})$ when the Taylor series is expanded around E(y). If $\hat{y}$ is unbiased for y, then the expected value of e_r is zero. The errors in stem curve models are generally so small that variances of the relative errors are very close to logarithmic error variances. If the size is considered to be fixed, then the expected value of the diameter D(u) is:

$$E[D(u)] = \exp\{\mu + \tfrac{1}{2}\,\mathrm{var}[d(u)]\}, \tag{61.2}$$

where μ is the fixed part of the model (41.3) and $\mathrm{var}[d(u)] = B(u) + W(u)$. The expected value of the volume, E(V), for a given size s is obtained from the regression equation (33.3). Owing to the close association between size and volume, the relative error $(V - \hat{V})/E(V)$ is near to $(V - \hat{V})/V$.

If the size is a normally distributed random parameter and the random stand and tree effects are normally distributed, then the conditional expectations of diameter D(u) and volume V are $\hat{D}(u)$ and $\hat{V}$, respectively, which are used in the denominator of e_r in (61.1). Also in the standard regression approach, where measured variables are assumed to be fixed, $E[D(u)] = \hat{D}(u)$, and $E(V) = \hat{V}$.

The comparison of different approaches using relative errors is problematic because the denominators are different in each case. For instance, the expected value of $(y - \hat{y})/\hat{y}$ is not exactly zero, if s in the stem form model is assumed to be fixed. If two or more dimensions of a tree are measured, the different relative errors are very close to each other.

According to the stem form model, the random variation of the stem form can be partitioned into variation between stands and within stands. Thus the prediction errors arise from these two sources of variation. Because the model works in the logarithmic scale, the partitioning of errors is most straightforward for the logarithmic or relative errors.

Relative errors for tree i in stand k are described by the following empirical model:

$$(y_{ki} - \hat{y}_{ki})/E(y_{ki}) = m + b_k + e_{ki}, \tag{61.3}$$

where m is the overall bias, b_k is a random error for stand k and e_{ki} is a random error

for tree i within stand k. Analysis of variance estimates (p. 474 in Searle 1971) are used to estimate $var(b_k)$ and $var(e_{ki})$; the estimates are denoted by s_b^2 and s_w^2, respectively.

The prediction errors can be partitioned into components by using our theoretical model for the stem form variation. For simplicity, the simple descriptive model presented is used to describe the prediction of stem curves. It can also be used for analyzing errors in the reference models. Theoretical total variances will, however, be compared with the empirical total variances in section 73. For the estimation of volume, theoretical variance components (described in section 53) will be compared with empirical results.

In typical applications the partitioning of the errors into components is important. Usually stem curve and volume equations are applied for several trees in a given stand, and the main interest lies in the mean (total) characteristics of the trees. According to the model (61.3), the mean of n relative errors in a stand has a mean square error (MSE)

$$m^2 + var(b_k) + var(e_{ki})/n. \qquad (61.4)$$

Thus the within-stand errors cancel each other as the number of trees increases, but the between-stand error and bias remain unchanged. Hence the main objective for a model is to produce small bias and between-stand error variance. If the between-stand error component is ignored in the error analysis (as is usually done in standard regression analysis), then the mean square error for a stand mean is underestimated. Note that MSE of the stand mean is not directly related to MSE of the mean of relative errors, if the size of trees in a stand varies greatly.

In the following applications the mean, standard deviation and root mean square error (RMSE) are also computed for the arithmetic errors. The mean square error of the arithmetic errors is computed as the sum of the squared mean and the variance of the errors. Sample variance of the arithmetic errors does not, however, correspond to or estimate (even approximately) any quantity in the stem form model.

62. Basic comparisons

Parameters of the model were estimated using Laasasenaho's data. Thus a good starting point for testing the model is to apply it in situations where the stem form and volume equations given by Laasasenaho (1982) can be used for comparison. In the tests all of the data was used, i.e., also those stands with one measured tree, which were not included in the analysis stage. Relative size of those single trees was given a value of 0.25, which seemed to be in accordance with their average stem form. The following equations of Laasasenaho (1982) define the reference models:

measured	stem curve equations	volume equation
$D_{1.3}$	(41.1)	(61.2)
$D_{1.3}$, H	(41.1) + (41.2)	(61.3)
$D_{1.3}$, H, D_6	(41.1) + (41.3)	(61.7)

According to Laasasenaho (1982), the total volume is calculated as the volume of the stemwood from the stump to the top of the tree. The stump is defined as being at the level of the uppermost root collar affecting cutting, or at least 10 cm. For integration of the stem volume using the predicted stem curve, the stump height is predicted using the regression equation (81.1) of Laasasenaho (1982) which has the diameter at breast height, $D_{1.3}$, and the height of the tree, H, as explanatory variables. If either of these values is not measured, it is replaced by the respective predicted value.

First applications of the stem form model are based on the overall model. The size of a tree is taken to be fixed; the random stand effects are estimated using the first four principal components of the between-stand covariance matrix **B**; and the stem curves of all trees belonging to the same stand are predicted simultaneously, i.e., the stand structure is utilized as it appears in the data. Later, if not indicated otherwise, the stem form model is also applied in the same way.

First consider the estimation of the stem volume using $D_{1.3}$. The error statistics for the arithmetic errors were for the stem form model (sfm) and for Laasasenaho's model (Laas.):

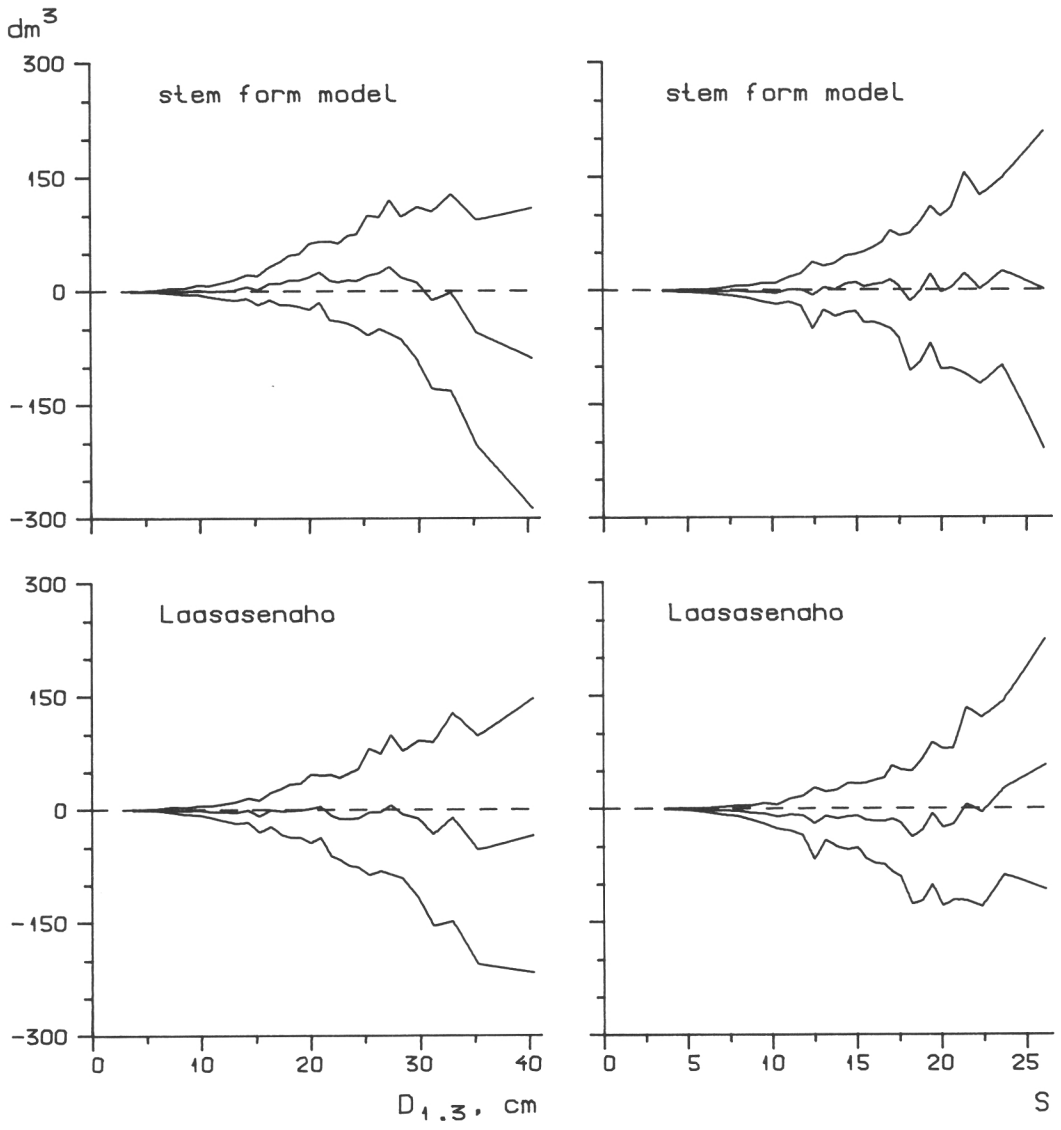

Fig. 12. Mean and mean ± standard deviation of $V - \hat{V}$ (dm^3) within groups of 80 trees (86 in the last group) with respect to $D_{1.3}$ and arithmetic size S when $\hat{V}$ is computed according to the stem form model or according to Laasasenaho's model.

model	mean dm^3	sd dm^3	RMSE dm^3	RMSE/$\bar{V}$ %
sfm	3.8	75.9	76.0	24.3
Laas.	—7.4	73.9	74.2	23.8

The above error figures are about equal. The principal difference between the two modelling approaches can be seen in Fig. 12: volume estimates based on the stem form model are virtually unbiased with respect to size, and Laasasenaho's estimates are nearly unbiased with respect to the measured $D_{1.3}$. The bias in the arithmetic scale has been corrected using var(d)—var($\hat{d}$) (Eq. 53.6), which should produce unbiased estimates of the volume for the given (unknown) size. As shown in Fig. 13, this bias correction has a very small effect. The bias correction for the case where $\hat{V}$ is unbiased for V given the measurements (using var($\hat{d}$—d), Eq. 53.7) would have a significantly greater effect, if applicable.

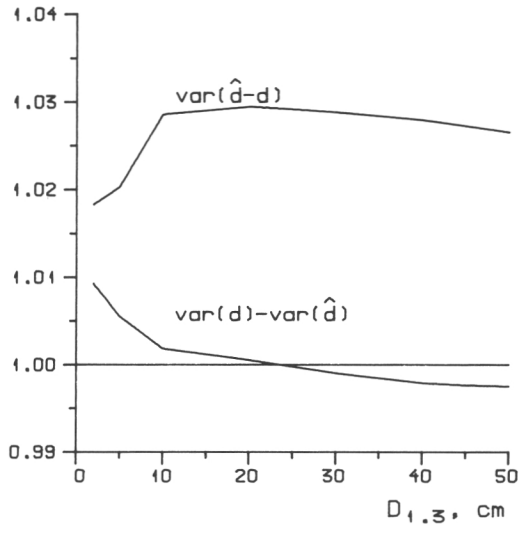

Fig. 13. The effect of the bias correction in volume estimation when only $D_{1.3}$ has been measured. The corrected estimate divided by the uncorrected estimate as a function of $D_{1.3}$. The bias correction for $\hat{d}$ is made by var(d)—var($\hat{d}$), which produces unbiased volume estimates for the given size, or by var($\hat{d}$—d), which would produce volume estimates unbiased for the fixed values of $D_{1.3}$, if $\hat{d}$ were a conditionally unbiased estimate of d for given $D_{1.3}$. The effect of the relative size is ignored.

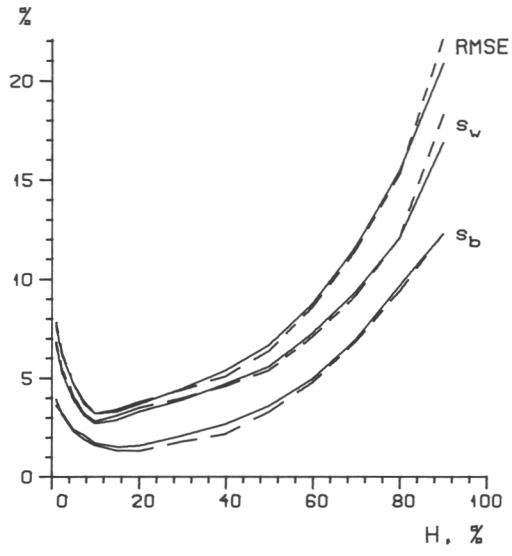

Fig. 14. s_b, s_w and RMSE of $(D—\hat{D})/\hat{D}$ at different relative heights for the stem form model (solid line) or for Laasasenaho's model (broken line) when $D_{1.3}$ and H have been measured.

Error statistics were also computed for relative errors defined by dividing $V—\hat{V}$ by $\hat{V}$, V or E(V) (Table 5). Consistent with the two modelling approaches, the stem form model is better with respect to relative error $(V—\hat{V})/E(V)$ (or $(V—\hat{V})/V$), and Laasasenaho's model is better with respect to error $(V—\hat{V})/\hat{V}$. As will be discussed later, $D_{1.3}$ is a better measurement for estimating the volume of small trees than bigger ones. Thus the relative errors for small trees are smaller than those for big trees: the root mean square error (RMSE) of relative errors is smaller than the RMSE/$\bar{V}$ of arithmetic errors.

In the second case we assume that in addition to $D_{1.3}$, the height, H, is also measured for each tree. Different relative errors are now very close to each other:

error	mean %	s_b %	s_w %	RMSE %
$(V—\hat{V})/E(V)$	0.1	3.9	6.3	7.4
$(V—\hat{V})/V$	0.0	3.8	6.2	7.3
$(V—\hat{V})/\hat{V}$	0.5	3.8	6.2	7.3

Table 5. Mean, between-stand standard error (s_b), within-stand standard error (s_w) and root mean square error (RMSE) of relative errors when the volume is estimated with measured $D_{1.3}$ using either the stem form model (sfm) or Laasasenaho's model. Relative error is defined by dividing $V—\hat{V}$ by the expected volume for the given size, E(V), by the true volume, V, or by the estimated volume, $\hat{V}$.

error	model	mean %	s_b %	s_w %	RMSE %
$(V—\hat{V})/E(V)$	sfm	0.5	15.8	11.8	19.8
	Laas.	—5.9	16.2	13.7	22.0
$(V—\hat{V})/V$	sfm	0.7	15.0	11.4	18.8
	Laas.	—5.8	15.6	13.4	21.4
$(V—\hat{V})/\hat{V}$	sfm	4.1	14.6	11.2	18.9
	Laas.	—2.1	13.3	12.0	18.0

Table 6. Error statistics, when $D_{1.3}$ and H are measured. Diameters at relative heights and volumes are predicted by the overall model (upper figures) and by Laasasenaho's model (lower figures). Mean, standard deviation (sd) and root mean square error (RMSE) are first given when errors are expressed in arithmetic units; RMSE is also given as a percentage of the overall mean. For relative errors ($(y-\hat{y})/\hat{y}$, in percentages) are given the mean, s_b, s_w and RMSE.

	arithmetic errors $y-\hat{y}$				relative errors $(y-\hat{y})/\hat{y}$			
H	mean	sd	RMSE	RMSE/$\bar{x}$	mean	s_b	s_w	RMSE
%	cm	cm	cm	%	%	%	%	%
1.0	0.11	1.95	1.96	7.2	0.4	3.6	6.7	7.6
	0.04	2.00	2.00	7.4	—.0.5	3.9	6.8	7.8
2.5	0.04	1.27	1.27	5.3	0.2	3.2	5.2	6.1
	—0.06	1.31	1.32	5.4	—0.6	3.1	5.4	6.2
5.0	0.12	0.71	0.72	3.3	0.6	2.4	3.9	4.7
	0.08	0.73	0.74	3.4	0.1	2.3	4.1	4.7
7.5	0.02	0.49	0.49	2.4	0.2	2.1	3.1	3.8
	0.02	0.50	0.50	2.4	0.0	1.9	3.2	3.7
10.0	0.01	0.46	0.46	2.3	0.1	1.8	2.7	3.2
	—0.05	0.48	0.48	2.4	—0.2	1.6	2.8	3.2
15.0	0.04	0.59	0.59	3.1	0.2	1.5	2.9	3.3
	—0.03	0.62	0.62	3.3	—0.1	1.3	3.1	3.4
20.0	0.02	0.72	0.72	4.0	0.1	1.6	3.3	3.7
	—0.01	0.73	0.73	4.0	0.0	1.3	3.5	3.8
30.0	0.06	0.85	0.85	5.0	0.3	2.1	3.9	4.5
	—0.03	0.84	0.84	5.0	—0.1	1.8	4.0	4.4
40.0	0.07	0.95	0.95	6.1	0.3	2.7	4.7	5.4
	—0.03	0.92	0.92	5.9	—0.2	2.2	4.6	5.1
50.0	0.06	1.03	1.03	7.3	0.4	3.6	5.6	6.7
	—0.02	1.01	1.01	7.2	—0.1	3.3	5.4	6.4
60.0	0.04	1.17	1.17	9.6	0.3	5.0	7.3	8.8
	—0.02	1.14	1.14	9.3	—0.2	4.8	7.1	8.6
70.0	0.00	1.28	1.28	12.9	0.2	7.0	9.4	11.8
	—0.03	1.22	1.22	12.3	—0.2	6.9	9.2	11.5
80.0	0.05	1.23	1.23	17.3	1.1	9.7	12.1	15.5
	0.00	1.14	1.14	16.1	0.2	9.4	12.1	15.3
90.0	0.06	0.93	0.93	24.9	2.0	12.4	16.9	21.1
	—0.01	0.85	0.85	22.7	1.9	12.3	18.3	22.1
Vol.	dm^3	dm^3	dm^3	%	%	%	%	%
	0.67	32.50	32.51	10.4	0.5	3.8	6.2	7.3
	—0.68	33.89	33.90	10.9	0.0	3.7	6.2	7.3

The prediction errors are so small that the bias correction for the arithmetic scale is negligible, as shown in the following set-up for error $(V-\hat{V})/E(V)$:

bias correction	mean %	RMSE %
no	—0.2	7.4
var(d)—var($\hat{d}$)	0.1	7.4
var($\hat{d}$—d)	—0.2	7.5

Error statistics are shown in Table 6 and in Fig. 14 for the stem form model and for Laasasenaho's models using error $(y-\hat{y})/\hat{y}$. The results are practically unbiased for both models, i.e., the mean of the errors does not contribute to the RMSE. Both models are about equally good in prediction of the stem curve as well as in volume estimation.

In the third application we further assume that the diameter at 6 m, D_6, is also mea-

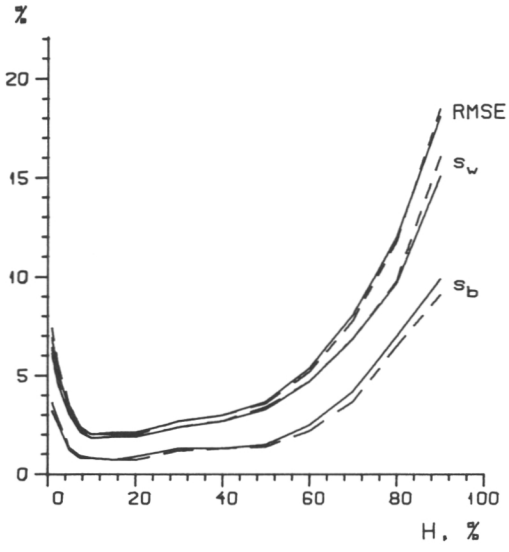

Fig. 15. s_b, s_w and RMSE of $(D-\hat{D})/\hat{D}$ at different relative heights for the stem form model (solid line) or for Laasasenaho's model (broken line) when $D_{1.3}$, H and D_6 have been measured. Only trees taller than 7 m are included.

sured. The stem form model and Laasasenaho's models are also in this case about equal. Fig. 15 shows s_b, s_w and RMSE of $(D-\hat{D})/\hat{D}$. In volume estimation, the error statistics for $(V-\hat{V})/\hat{V}$ were as follows:

model	mean %	s_b %	s_w %	RMSE %
sfm	—0.4	1.6	3.2	3.6
Laas.	0.0	1.5	3.2	3.5

The stem form model does not predict well the upper parts of the stem curves of very exceptional trees. This seems to be an inherent property of the polar coordinate system: the angle between two points on the stem curve is not always as good a measure of the distance as the usual height difference. For instance, for a thick tree the distance between the D_6-point and the top of the tree can be very long when measured in angles (and small as a height difference); therefore the model does not force the diameter down to zero rapidly enough when moving from the D_6-point upwards.

In conclusion, when $D_{1.3}$, H and possibly D_6 are measured, the stem form model gives about as good results as the special-purpose equations of Laasasenaho (1982). However,

Table 7. Mean, between-stand standard deviation (s_b), within-stand standard deviation (s_w), and root mean square error (RMSE) of relative errors of volume estimates, $(V-\hat{V})/E(V)$, for different measurement combinations. If D_6 is included, only trees taller than 7 m have been accepted.

measured	mean	s_b	s_w	RMSE
$D_{1.3}$	0.5	15.8	11.8	19.8
$D_{1.3}$, H	0.1	3.9	6.3	7.4
$D_{1.3}$, H, H_c	0.1	3.9	6.3	7.4
$D_{1.3}$, H, D_6	—0.5	1.6	3.3	3.7
$D_{1.3}$, H, D_6, H_c	—0.5	1.4	3.2	3.6

the stem form model is computationally more difficult to apply. The volume equations of Laasasenaho are not compatible (see Demaerschalk 1972) with his stem form functions, but as indicated by Table 10 on p. 53 in Laasasenaho (1982), his stem curve models give almost as good volume estimates as his volume equations do.

The usefulness of the crown height measurements (H_c) is studied by estimating stem volumes with measurement combinations ($D_{1.3}$, H, H_c) and ($D_{1.3}$, H, D_6, H_c). Relative errors of the volume estimates ($(V-\hat{V})/E(V)$) are summarized in Table 7 for all measurement combinations used. The crown height improves volume estimates only slightly. However, as will be seen later, the crown height may be a useful measurement in calibration of the stem curve.

63. Differences between climatic regions

Basic comparisons with the equations of Laasasenaho (1982) were based on the overall model; Laasasenaho also estimated the parameters of his equations for the whole country. In the following discussion, the regional differences will be studied using error $(V-\hat{V})/E(V)$.

Table 8 presents the regional means and summary statistics of the relative errors for different combinations of measurements when the overall model or the regionalized model is used for prediction. Let us first discuss the case in which only $D_{1.3}$ is measured. The regional means of the relative errors are rather large for the overall model, but their influence in RMSE is modest. If the regionalized version of the model is

Table 8. Regional means ($\bar{e}_i$), grand mean ($\bar{e}_.$), between-stand standard deviation (s_b), within-stand standard deviation (s_w) and RMSE of the relative volume errors (relative to E(V)) for the overall model (o.a.) and the regionalized model (reg.), and for different measurement combinations.

	measured						
	$D_{1.3}$		$D_{1.3}$, H			$D_{1.3}$, H, D_6	
	model						
	o.a.	reg.	o.a.	reg.		o.a.	reg.
	n				n		
$\bar{e}_1$	112	—9.5	—1.7	4.0	—0.4	105	—1.3	—0.8
$\bar{e}_2$	234	—0.3	—0.3	3.4	—0.1	206	—1.0	—1.0
$\bar{e}_3$	199	0.2	—1.1	2.5	—0.1	182	—0.6	—0.6
$\bar{e}_4$	429	2.8	1.1	0.0	0.2	354	—0.4	—0.3
$\bar{e}_5$	616	5.5	0.7	—1.1	0.0	559	—0.6	—0.6
$\bar{e}_6$	440	6.3	0.4	—2.5	—0.1	414	—0.2	—0.2
$\bar{e}_7$	22	—6.3	—3.9	—7.1	0.1	22	—0.4	—0.4
$\bar{e}_8$	274	—18.4	—1.1	1.5	—0.1	177	—0.2	—0.4
$\bar{e}$		0.5	0.1	0.1	0.0		—0.5	—0.5
s_b	2326	15.8	12.7	3.9	3.4	2019	1.6	1.6
s_w		11.8	11.3	6.3	6.2		3.3	3.2
RMSE		19.8	17.0	7.4	7.1		3.7	3.6

used, the regional means (biases) are insignificant (compared with s_b), except for the small and strange coastal region 7. As the regional biases disappear, s_b, the standard deviation of the between-stand errors, is also decreased. The regionalized model does not have any significant effect on the standard deviation of the within-stand errors, s_w.

When the height of each tree is also measured, the same qualitative conclusions can be made. Now the regionalized model removes the bias of region 7. If, in addition to H and $D_{1.3}$ also D_6 is measured, the regionalized model improves the results only slightly. Since the regional biases are already quite modest for the overall model, the regionalized model cannot contribute much.

The regional parameters represent our prior knowledge of the stem form in different regions. If only one or two dimensions of a tree are measured, this regionalized prior knowledge significantly improves the prediction accuracy. Comparison of Figures 4 and 7 shows that the regional differences are not solely in the thickness component, which can already be estimated quite well with two measurements. If three or more dimensions are measured, the regionalized prior knowledge contains little new information.

The regionalized model was also tested in the case where the crown height is measured. The influence of the crown height was similar to the results for the overall model presented in Table 7.

64. Effect of grouping trees into stands

The stem form model is based on an analysis of variation in stem form of stands and of the trees within stands. The stand structure enters into the model through the random stand effects and through the average (relative) size. In this section we examine how significant these stand effects are in prediction. Assume first that the same measurements are made for each tree; use of mixed measurement combinations is referred to in the next section.

To reveal the effect of grouping trees into stands, the stem curves and stem volumes were predicted ignoring the stand structure, i.e., the prediction was made for each tree separately. The results were then compared with the earlier results obtained when stem curves were predicted simultaneously for all trees belonging to the same stand.

When $D_{1.3}$ alone was measured, there were differences between simultaneous and separate prediction, as shown below for the error $(V-\hat{V})/E(V)$:

estimation	mean %	s_b %	s_w %	RMSE %
separately	—0.2	15.1	14.7	21.1
simultaneously	0.5	15.8	11.8	19.8

All differences between the simultaneous and separate prediction are due to the relative size, because with only one measured dimension the estimates of the random

stand effects are zero if the prediction is made separately or simultaneously. The increase in RMSE in the separate prediction is due to the increase in the variance of within-stand errors. This may be explained as follows. In a separate prediction the relative size of each tree is estimated to be zero; therefore the sum of relative sizes of all trees in a stand is also zero. According to the stem form model, the logarithmic diameter at a given angle is a linear function of the relative size. If a diameter is measured at the same angle for each tree, then the errors caused by taking all the relative sizes to be zero cancel each other when added over the stand. Angles corresponding to the diameters at breast height are usually quite similar for trees in the same stand.

When two or more dimensions are measured, then not only the relative size effect but also the estimation of random stand effects is different, depending on whether trees are grouped into stands or not. Estimates of random stand effects are also obtained when two dimensions of a single tree are measured. With several trees these estimates merely become more accurate. When two or more dimensions were measured, however, the grouping of trees into stands did not improve the predictions. This calls for an explanation both with respect to the relative size and the random stand effects.

The relative size of a tree only provides information about the thickness component of the stem form, as discussed previously. If two or more dimensions are measured from different parts of the stem, however, the thickness component of the tree can be determined accurately. Thus the relative size looses its significance in the prediction of stem form.

The fact that the more accurate estimates of stand effects are insignificant may be explained as follows. When the same, or closely related, dimensions are measured for several trees in a stand, we get more accurate information about the stand average of these and correlated dimensions. In the next section we will see that, if we have not measured these dimensions for a tree, such information helps to predict them. On the contrary, if we have measured any dimension, then knowledge about the stand average of the same dimension does not give any new information. In practice, the above conclusions are also valid for diameters at a given height even if in the polar coordinate system these diameters are not same dimensions for different trees.

65. Standwise calibration of the stem curve

One promising application of the stem form model is the standwise calibration of the stem curve. The stem curve is calibrated by estimating the random stand effects. As shown in the previous section, calibration does not help if the same dimensions are measured for each tree. In this section we consider the case in which different measurements are made for different trees. The stem form model can be applied for any mixture of measurements, but for simplicity (and following the usual practice) trees in a stand are grouped into two sets, calibrating trees and tally trees, to which the calibrated stem curve is applied. Different combinations of measurements are used for calibrating trees; $D_{1.3}$ is assumed to be measured for tally trees. In forestry, calibrating trees are called 'sample trees'. Here only volume estimation will be considered; the calibration of the height curve is illustrated in Fig. 16.

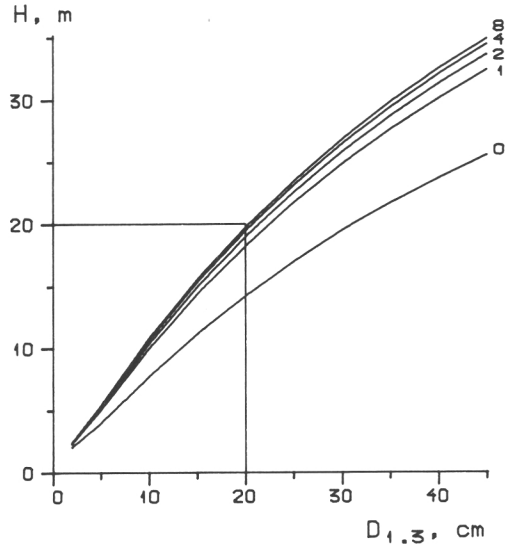

Fig. 16. Calibration of the height curve in a stand. The predicted height as a function of the measured $D_{1.3}$ when there are 0, 1, 2, 4 or 8 calibrating trees with $D_{1.3}$=20 cm and H=20 m; the effect of the relative size is ignored.

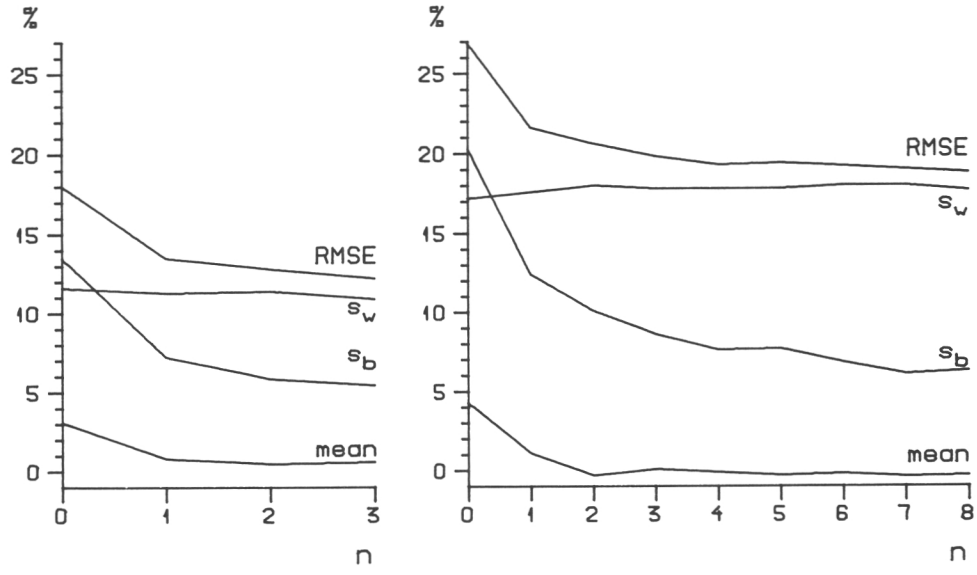

Fig. 17. Mean, between-stand standard deviation (s_b), within-stand standard deviation (s_w) and RMSE of the volume estimation errors of tally trees as a function of the number of calibrating trees; calibrating measurements are $D_{1.3}$ and H; the overall model. Data of Laasasenaho (left) and data of Kilkki and Varmola (right).

When computing the test results of this section, calibrating trees were first chosen randomly, and all other trees in the stand were used as tally trees. Then the random stand effects were estimated using the calibrating trees. The size parameters and volumes of tally trees were estimated using the estimated stand effects. Relative errors in volume estimation are analyzed with the variance components model (61.3). Estimates of the error variance components differ if different calibrating trees are selected. Thus error variance components were estimated 30 times with different seed values in the random number generator used to select the calibrating trees. The final estimates of variances are then averages of the 30 preliminary estimates.

The error variance components s_b^2 and s_w^2 can be estimated only if there are stands with at least two trees. Hence, because there are at most 5 trees per stand in the data, the error variance components for the tally trees can be estimated only if there are at most 3 calibrating trees. In order to have the same data in each case, only stands with 4 or 5 trees were used all the time. Because the trees have been sampled with a relascope, the stands in the tests have a larger basal area than do stands on the average. Thus the subdata used deviates systematically from the whole data. Calibration was also studied using the data of Kilkki and Varmola. With their data there can be 8 trees in the calibrating set, and 28 of the total 29 stands can still be utilized.

In the first application we assume that $D_{1.3}$ and H are measured for the calibrating trees. Fig. 17 shows the mean, between-stand standard deviation (s_b), within-stand standard deviation (s_w) and RMSE of the relative errors $(V-\hat{V})/E(V)$ with respect to the number of calibrating trees in the data set of Laasasenaho and of Kilkki and Varmola.

The initial bias in the data of Laasasenaho shows that the subdata used deviate from the whole data. In the whole data the model is practically unbiased (see Table 5); note also that initially (without calibrating trees) s_b is smaller in the subdata than in the whole data. In both data sets the bias and s_b decrease rapidly.

The within-stand standard error s_w is practically constant, as it should be. In the data of Kilkki and Varmola both s_b and s_w are clearly higher than in the data of Laasasenaho. The calibration seems to work well also for their data.

Table 9. $RMSE_b$ for different calibrating measurement combinations.

calibrating measurements	data of Laasasenaho							
	overall model				regionalized model			
	number of calibrating trees							
	0	1	2	3	0	1	2	3
$D_{1.3}$, H	13.8	7.3	5.8	5.3	11.1	6.7	5.4	5.2
$D_{1.3}$, H, H_c	13.8	7.1	5.8	5.3	11.1	6.5	5.1	4.9
$D_{1.3}$, H, D_6	13.8	7.2	5.9	5.3	11.1	6.8	5.5	5.2
$D_{1.3}$, H, D_6, H_c	13.8	7.4	5.9	5.4	11.1	6.5	5.2	5.0

	data of Kilkki and Varmola overall model								
	number of calibrating trees								
	0	1	2	3	4	5	6	7	8
$D_{1.3}$, H	19.6	13.1	9.9	8.8	7.8	7.8	6.7	6.3	6.3
$D_{1.3}$, H, H_c	19.6	12.9	9.6	8.5	7.6	7.5	6.3	5.9	5.9
$D_{1.3}$, H, D_6	19.6	13.0	9.8	8.5	7.9	7.8	6.4	6.5	6.2
$D_{1.3}$, H, D_6, H_c	19.6	12.8	9.6	8.3	7.6	7.5	6.2	6.1	5.8

The results are similar for the other tested combinations of calibrating measurements. The between-stand RMSE ($RMSE_b$) combining the bias and between-stand standard error is presented in Table 9. The results for different combinations of calibrating measurements are very close to each other. Theoretical analysis shows later that the differences become clearer as the number of calibrating trees increases. Combination ($D_{1.3}$, H, H_c) seems to be better than ($D_{1.3}$, H, D_6), but later we will see that this probably holds true only for a small number of calibrating trees.

For the calibrating measurement combination ($D_{1.3}$, H, D_6), the stem form model can be compared with the calibration system of Pekkonen (1983). Pekkonen uses a polynomial regression predictor to estimate stem volume as a function of $D_{1.3}$. The coefficients of the predictor are calibrated by measuring $D_{1.3}$, H and D_6 for the calibrating trees. The volumes of the calibrating trees are first estimated with Laasasenaho's model; the estimates are assumed to give the volume correctly. Then the diameter-volume data of the calibrating trees are added to the a priori data using a heuristic weighting procedure. Pekkonen's system and the stem form model are compared using $RMSE_b$. According to Pekkonen's system, only trees with $D_{1.3}$ between 4 and 44 cm have been used in the comparison. Pekkonen assumes that the height of the measurement is determined from the uppermost root collar. In the stem form model, the same diameters were used; but the height is measured from the ground. $RMSE_b$ decreases much faster initially with the stem form model than with Pekkonen's system (Fig. 18). However, when the number of calibrating trees increases the difference between these systems becomes smaller.

Further comparisons with Pekkonen's system are made using the data of Pekkonen and Laasasenaho, in which each stand included at least 51 pine stems. We can assume that the total (stand) volume of tally trees is not affected by the tree effects, if the number of tally trees is moderate (e.g. ≥ 20). Thus the stand errors can be studied directly using stand totals, without a variance components model. This way, large trees influence the results more than in the earlier comparisons.

The model was estimated using trees sampled with a relascope, i.e., the sampling probabilities were proportional to the basal area (or $D_{1.3}^2$). If the calibrating trees are selected similarly, their selection is in accordance with the estimation of the model parameters. The methods are compared using the mean, standard deviation and RMSE of relative stand errors $(V_{k.} - \hat{V}_{k.})/\hat{V}_{k.}$ (Fig. 19). Errors $(V_{k.} - \hat{V}_{k.})/V_{k.}$ are slightly smaller for both calibration methods, the result being the same: Pekkonen's method is better if the number of calibrating trees is greater than 10. There are two possible explanations for this.

First, the dependence of stem form on size may vary slightly from stand to stand.

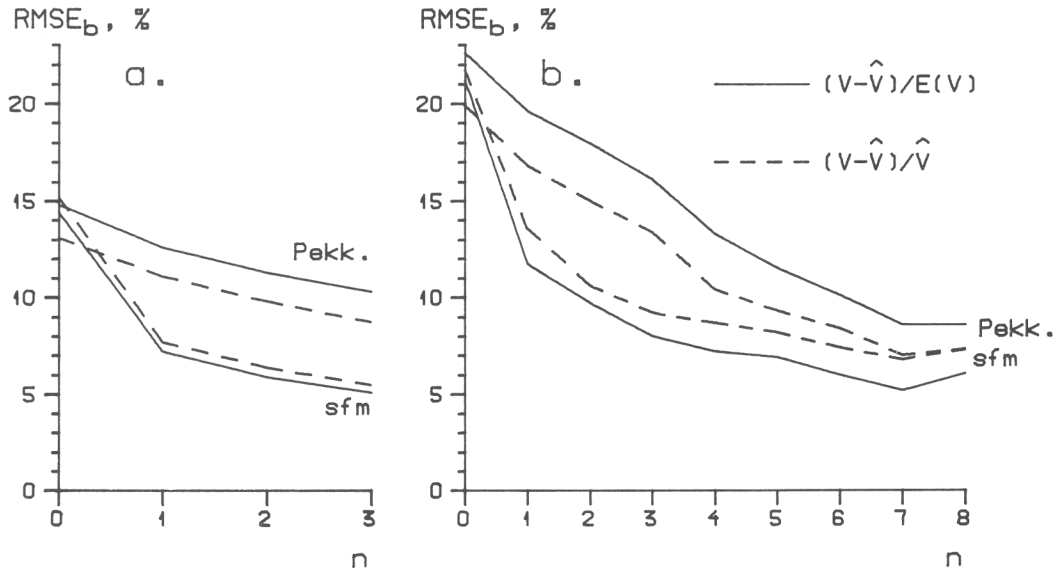

Fig. 18. $RMSE_b$ as a function of the number of calibrating trees both for error $(V-\hat{V})/E(V)$ (solid line) and for error $(V-\hat{V})/\hat{V}$ (broken line); the calibration is done according to the system of Pekkonen (Pekk.) or with the stem form model (sfm, the overall model). Data of Laasasenaho are used in Fig. 18a and data of Kilkki and Varmola in Fig. 18b.

This is taken into account by Pekkonen's method but not by the stem form model. Although the stem form model seems to reveal most of the variation in the stand stem curves, it is not flexible enough to describe each stand in detail.

Second, height (total height or the height of a diameter) is measured from the uppermost root collar in Pekkonen's system and from the ground in the stem form model. As the volume is determined from the uppermost root collar (except for small trees), Pekkonen's system has thus an advantage over the stem form model. If, for the stem form model, the height from the ground was computed by adding the estimated height of the root collar to the measured height from the root collar, the standard deviation of relative stand errors became smaller than in Pekkonen's method. However, owing to an increase in the bias, the RMSE increased slightly. The reason for this additional bias is probably the irregularity of the uppermost root collar in the data of Pekkonen and Laasasenaho as compared with the data of Laasasenaho. In any case, the stem form model can be recommended when only a few calibrating measurements are made.

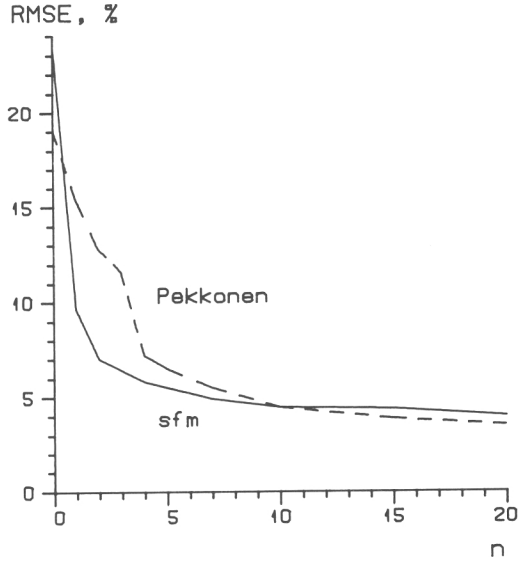

Fig. 19. RMSE of relative stand errors $(V_{k.}-\hat{V}_{k.})/\hat{V}_{k.}$ as a function of the number of calibrating trees using the stem form model (sfm) or the system of Pekkonen; the data of Pekkonen and Laasasenaho.

Calibrating measurement combinations ($D_{1.3}$, H) and ($D_{1.3}$, H, D_6) were also compared using relative stand errors in the data of Pekkonen and Laasasenaho (Fig. 20). The upper diameter D_6 was worth of measuring if the number of calibrating trees was greater than three.

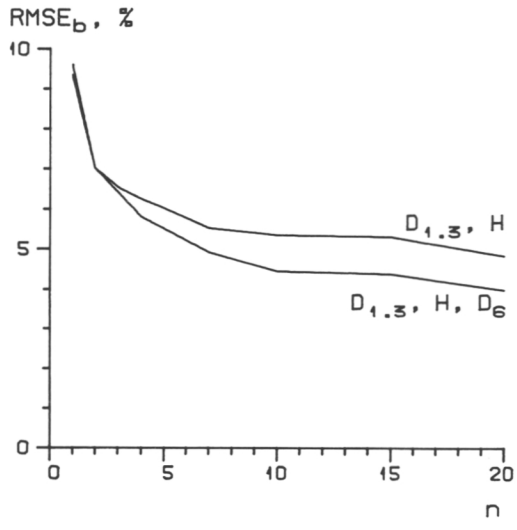

Fig. 20. RMSE of relative stand erros $(V_{k.}—\hat{V}_{k.})/\hat{V}_{k.}$ as a function of the number of calibrating trees for calibrating measurements ($D_{1.3}$, H) and ($D_{1.3}$, H, D_6); the data of Pekkonen and Laasasenaho.

7. FURTHER APPLICATIONS

71. Size as a random parameter

In this chapter we study applications based on the knowledge of different distributions. In the first two sections, the distributions of the size parameter and measurement errors will be discussed. The last three sections deal with the estimation of the error variances and their application in timber assortment problems and optimization of measurements.

In principle, there are two main ways in formulating and utilizing the prior knowledge to make the predictions of the stem form model more accurate. The first method is exemplified in the regionalized model: we can divide the domain of interest into subdomains defined by easily measurable characteristics, and then estimate different parameters for different subdomains. The division into subdomains can even be continuous: the parameters can be defined as functions of some continuous external variables.

The second way to formulate a priori knowledge is in the form of prior distributions for some parameters, i.e., we can assume certain parameters to be random. The stand effect, v(u), has been a random parameter from the beginning. In addition, the size of a stem may be assumed to be a random parameter. The proper distribution of size depends on the application at hand. Thus the size distribution as exemplified in the present data may not be useful in specific applications.

In order to apply the standard theory of linear prediction, the mean and variance of the random parameters must be known or assumed. In the data used in this study, the mean size is 2.59 (corresponding to a volume of 157 dm^3) and the standard deviation 0.48. The frequency histogram is presented in Fig. 21 with the corresponding normal density function. The size distribution is not exactly normal: the best linear unbiased predictor of size is not the best (minimum variance) unbiased predictor. Note that, in the case of

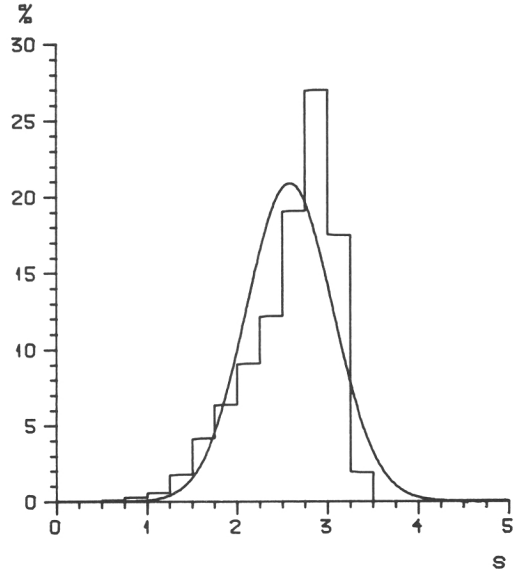

Fig. 21. Frequency histogram of size s in the data of Laasasenaho, and the corresponding normal density function.

random size, there are no fixed parameters in the model, hence a predictor is (unconditionally) unbiased if its expectation is equal to the population mean.

The effect of defining the size as a random parameter was tested for measurement combinations (H), ($D_{1.3}$), (H, $D_{1.3}$), and (H, $D_{1.3}$, D_6), using the mean and variance as observed. The sizes of different trees were assumed to be uncorrelated. In practice, and also in the data used, sizes of trees in the same stand are correlated, although this correlation was not taken into account. The bias correction for the volume estimates in the arithmetic scale was computed according to (53.7), i.e., using $\mathrm{var}(\hat{d}-d)$.

Let us first consider the case in which $D_{1.3}$ is the only measured dimension. If size is treated as a random parameter, then its estimates (and volume) are shifted towards

the assumed mean. In Fig. $\hat{V}^{1/3}$ is presented as a function of $D_{1.3}$ where $\hat{V}$ is estimated by means of Laasasenaho's model or using the stem form model for single trees (ignoring the relative size) with fixed or random size. When the estimation is done simultaneously for all trees in a stand, i.e., the effect of the relative size is taken into account, the volume estimates of trees with large $D_{1.3}$ would be closer to the estimates of Laasasenaho than those presented in Fig. 22.

As explained earlier, the stem form model gives volume estimates that are biased upwards for trees with large $D_{1.3}$, when s is treated as a fixed parameter. With random s, the volume estimates change only slightly and remain biased for trees with large $D_{1.3}$. The reason is that the distributions of stand effects, tree effects, and size are all skew. The skewness affects the results in the same direction in each case: there are more thick stands and trees than thin ones (Fig. 8), and there are less big trees than small ones (Fig. 21). Hence, if $D_{1.3}$ is measured alone (or together with H and D_6), no difference appears in the RMSE of volume estimates.

When height is the only dimension measured, then in principle the situation is the same as with measured $D_{1.3}$. The variances of the random effects of height are, however, larger (Table 2), and so the prior distribution of size has more weight in the estimation. Thus the RMSE of the arithmetic errors decreased with random size from 211 dm^3 to 171 dm^3, and RMSE of $(V-\hat{V})/\hat{V}$ changed from 81 % to 65 %. Consistently, RMSE of $(V-\hat{V})/E(V)$ increased from 51 % to 62 %. These error figures are much larger than those observed when $D_{1.3}$ is measured alone. Thus the height is not usually measured alone, and this case has only theoretical interest.

The randomness of the size parameter was then studied further using a more heuristic approach. As noted, the proper size distribution is different for different applications. We can, however, have general information about how big Scots pines can grow in Finland. The relative frequencies of big trees are much greater in the data than in the forests generally. Thus the deep right tail of the size distribution (Fig. 21) shows that trees with s greater than 3.5 are exceptionally rare in Finland.

If the overall mean and variance of any general tree set are used to determine the prior mean and variance of the random size, then the standard linear models do not shrink the size estimates for big trees enough towards the mean. Moreover, the size estimates for the small trees are increased without any theoretical justification.

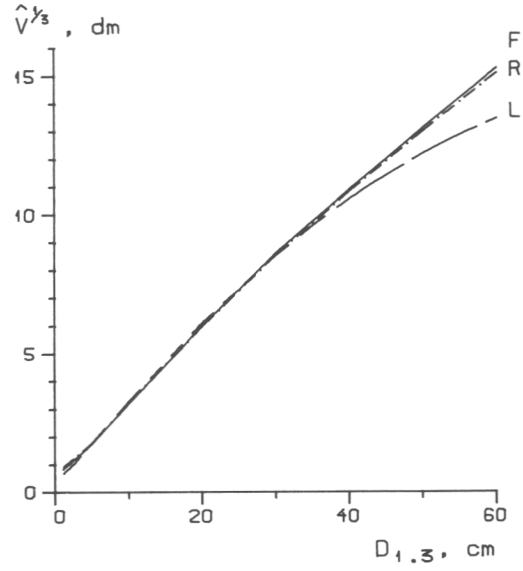

Fig. 22. $\hat{V}^{1/3}$ as a function of $D_{1.3}$. $\hat{V}$ is computed with Laasasenaho's model (L), or using the stem form model with fixed size parameter (F), or with random size parameter (R, mean of s is 2.59 and standard deviation 0.48). In the stem form model the effect of the relative size is ignored. The maximum of $D_{1.3}$ is 50.9 cm in the data of Laasasenaho.

The one-sided prior information about the size distribution was incorporated into the model using the following heuristic method. First the preliminary estimate of size ($\dot{s}$) was computed as usual. If $\dot{s}$ was smaller than a given value μ_s, then the size was treated as a fixed parameter. If $\dot{s}$ was greater than μ_s, then s was treated as a random parameter with mean μ_s and standard deviation σ_s. With this method, $\hat{V}$, estimated using $D_{1.3}$, can be made unbiased for large values of $D_{1.3}$. For instance, with $\mu_s = 3$ and $\sigma_s = 0.2$, the mean of arithmetic errors for trees with $D_{1.3}>35$ cm (N=133) was reduced from -84 m^3 to —9 dm^3 and RMSE from 184 dm^3 to 174 dm^3. The resulting predictor approaches Laasasenaho's model (see Fig. 22). Owing to the small number of big trees, the error statistics for the whole set of data did not improve noticeably.

In conclusion, some gain may be attained by defining the size as a random parameter, even if specification of the proper distribution is problematic. If the model is used to estimate volumes of big trees, and only one measured dimension is used, the heuristic approach presented may prevent the results from being biased with respect to the measured dimension.

If the size of a tree is partitioned into a random stand effect and a random tree effect, then the results will probably improve slightly. In this case, however, we should be able to specify two size distributions instead of one. The randomness of size may require a more sophisticated (Bayesian) analysis than was attainable in this study.

72. Simulations with measurement errors

The effect of the measurement errors can be studied by generating artificial random errors for the measurements. Section 54 described how the existence of the measurement errors can be taken into account in the estimation by adding the measurement error variances to the within-stand variances and how variances of the measurement errors should be transformed for the polar coordinate system.

As noted previously, errors in the diameter measurements have two different components: error due to the noncircular form of the stem and the pure measurement error. Errors due to the uncertain determination of the ground level are classified here as pure measurement errors. The first component can, in principle, be described by general models. In contrast, pure measurement errors are dependent on measuring devices, measuring conditions, and on human factors. Thus pure measurement errors are always dependent on the particular situation. Note that the height measurements contain only pure measurement errors.

For this study, the variances of the measurement errors were taken from the study of Hyppönen and Roiko-Jokela (1978). To simplify the treatment, observed biases were combined with the error variances, i.e., the observed mean square error was used to estimate the error variance, and the measurements are assumed to be unbiased. In the data of Hyppönen and Roiko-Jokela the sample standard deviations (including the biases) are in arithmetic units and in proportion to the mean values:

measurement	sd	sd/$\bar{x}$
$D_{1.3}$	0.3 cm	1.4 %
D_6	0.8 cm	5.3 %
H	0.9 m	6.3 %

Hyppönen and Roiko-Jokela (1978) found no clear connection between the error variance of height measurements and the height. They did not consider the dependence between the error variance of the diameter measurement and the true diameter.

In the simulations both arithmetic errors (fixed error variance) and relative measurement errors (variance proportional to the square of the measured dimension) were used. Arithmetic errors were generated according to the normal distribution and relative errors according to the lognormal distribution. As the relative errors are used to evaluate the volume estimates, arithmetic measurement errors have greater effect in small trees than in large ones. The effect of the measurement errors was computed both by taking the measurement error variances properly into account in the estimation procedure and by ignoring them. For the first case, the measurement errors are said to be 'corrected'. The error statistics are obtained as averages of 30 runs with different seed values for the random number generator. Error statistics are presented in Table 10 for measurement combinations ($D_{1.3}$), ($D_{1.3}$, H) and ($D_{1.3}$, D_6, H).

If only $D_{1.3}$ is measured and s is considered to be fixed, then the measurement errors do not alter the way the stem curve is predicted in logarithmic units. When predicted diameters are transformed from the logarithmic scale to the arithmetic scale, the increase in predictor variance changes the predicted stem curves slightly. This change has no practical effect: the volume estimates cannot be improved by taking the measurement errors into account. The estimates of the error variances will, however, improve. The within-stand errors of the volume estimates increase due to the measurement errors. If only one dimension (e.g. $D_{1.3}$) is measured for each tree, the between-stand standard error of the volume estimates does not change.

For other measurement combinations, measurement errors also increase s_w. If the

measurement errors are taken into account in the estimation, then the effect of the measurement errors on s_w can be decreased to some extent. In general, measurement errors increase s_b slightly.

The effect of measurement errors in the calibration of the stem curve was also studied. For the calibration we are interested in how errors in the calibrating measurements affect between-stand variance of the volume estimates for the tally trees. $D_{1.3}$ and H were assumed to be measured for the calibrating trees. Although measurement errors make the calibration slower, it is still quite effective (Fig. 23). If measurement errors are taken into account in the estimation, their impact can, to some extent, be compensated. Bigger errors are easier to correct. Thus the correction procedure is more efficient for arithmetic errors of measurement, which are large (in relative units) for small trees.

The effect of measurement errors on the calibration was also studied using measurement combination ($D_{1.3}$, H, D_6). The results were nearly the same as those in Fig. 23.

Table 10. Volume estimation when measurements are disturbed by simulated measurement errors. The variance of the measurement errors is first proportional to the square of the measured dimension, in which case the relative measurement error is 1.4 % for $D_{1.3}$, 5.3 % for D_6 and 6.3 % for H. In the second case ('arithmetic measurement error') the variance of the measurement errors is constant, the standard deviation being 0.3 cm for $D_{1.3}$, 0.8 cm for D_6 and 0.9 m for H. The measurement errors are either taken into account ('corrected') in the estimation or ignored.

error type		no error	rel.	rel.	arith.	arith.
corrected		—	no	yes	no	yes
measured						
$D_{1.3}$	mean	0.5	0.4	0.4	0.3	0.3
	s_b	15.8	15.9	15.9	15.9	15.9
	s_w	11.8	12.3	12.3	13.2	13.2
	RMSE	19.8	20.1	20.1	20.6	20.6
$D_{1.3}$,H	mean	0.1	—0.3	0.0	—0.2	—0.6
	s_b	3.9	3.7	4.2	4.4	4.5
	s_w	6.3	9.4	8.6	10.7	10.0
	RMSE	7.4	10.1	9.5	11.6	11.0
$D_{1.3}$,H,D_6	mean	—0.5	—1.0	—0.4	—1.0	—1.0
	s_b	1.6	2.1	2.1	2.0	2.1
	s_w	3.2	7.7	6.7	7.5	6.9
	RMSE	3.7	8.1	7.0	7.8	7.3

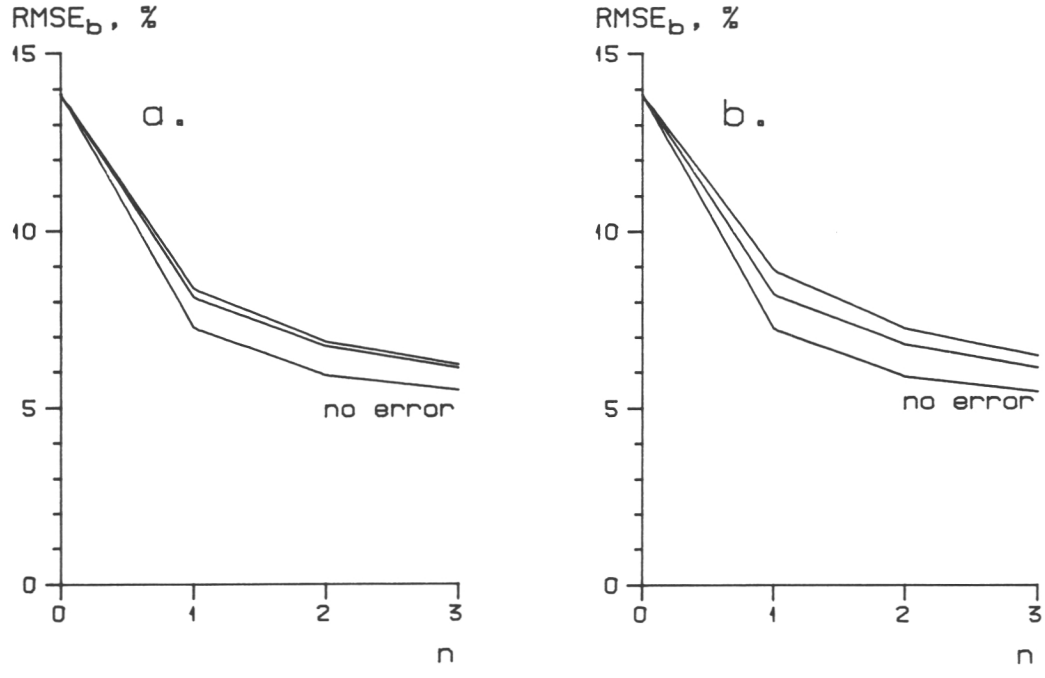

Fig. 23. Effect of errors in calibrating measurements. $RMSE_b$ of errors $(V—\hat{V})/E(V)$ as a function of the number of calibrating trees when there are no errors in the calibrating measurements (lowest line), are corrected measurement errors (middle line), or are uncorrected measurement errors (upper line). In Fig. 23a measurement errors have fixed variance in the logarithmic scale and in Fig. 23b in the arithmetic scale.

73. Estimation of error variances

In the stem form model we get estimates of the error variances for diameters at knot angles and for stem volume. Error variances in the polar coordinates can be transformed to error variances of diameters at a given height using the correspondence derived in section 54 for measurement errors. Estimates of error variances are useful in several applications, e.g., for estimating proportions of different timber assortments or for optimizing measurement strategies. In this section theoretical variances are compared briefly with the empirical (sample) error variances.

Error variances of the predicted stem curves in the polar coordinates are first compared in the logarithmic scale when either $D_{1.3}$ or $D_{1.3}$ and H are measured. Error variances vary from tree to tree, but for each stand k and tree i the expected value of $[d_{ki}(u)-\hat{d}_{ki}(u)]^2$ is equal to $\text{var}[d_{ki}(u)-\hat{d}(u_{ki})]$. Thus the average of the squared residuals (sample mean square error) should be approximately equal to the average of the theoretically computed variances. This equality should also hold for square roots, which can be interpreted as relative standard errors. As can be seen from Fig. 24, theoretical estimates of variance agree quite well with the empirical results.

Error variance for diameter predicted at a specified height can be estimated as follows. First, the error variances for the knot angles are expressed in terms of the height using the predicted stem curve. The variance for the given height is then interpolated with a cubic spline. This interpolated variance is divided by f^2, where f is the (interpolated) coefficient (54.9) that transforms an error in the height-diameter coordinates to polar coordinates as described in section 54. Fig. 25 shows the correspondence between the theoretical and empirical RMSE for different relative heights, when $D_{1.3}$ and H have been measured. Theoretical and empirical results agree quite well.

Variance components of volume estimates were derived in section 53. Empirical estimates of these variances are computed using the descriptive model (61.3) for relative errors $(V-\hat{V})/E(V)$. Observed biases are again combined with estimated between-stand variances. As can be seen from Table 11, the theoretical standard errors reflect well the

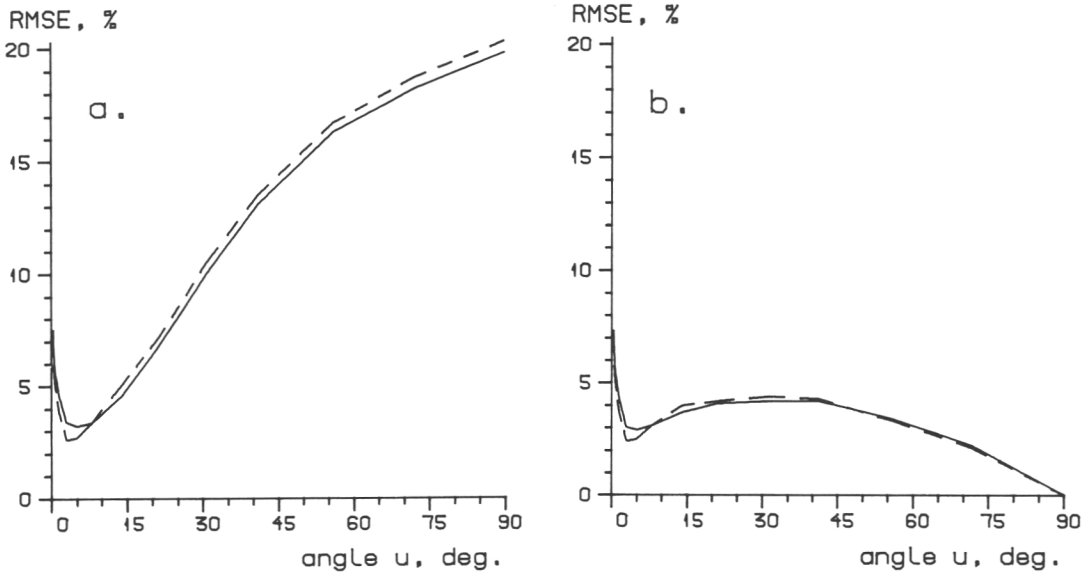

Fig. 24. Sample RMSE of predicted logarithmic diameter (solid line) compared with the theoretical counterpart (broken line) for different angles; $D_{1.3}$ is measured in Fig. 24a; $D_{1.3}$ and H are measured in Fig. 24b. Data of Laasasenaho, overall model.

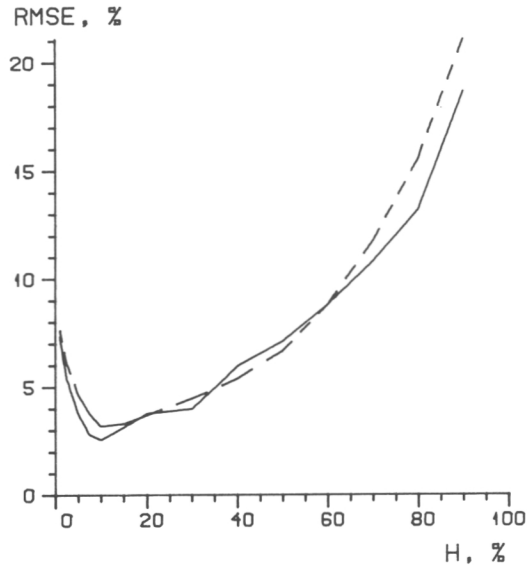

Fig. 25. Sample RMSE of relative errors $(D-\hat{D})/\hat{D}$ (solid line) compared with the theoretical counterpart (broken line) for different relative heights, when $D_{1.3}$ and H have been measured. Data of Laasasenaho, overall model.

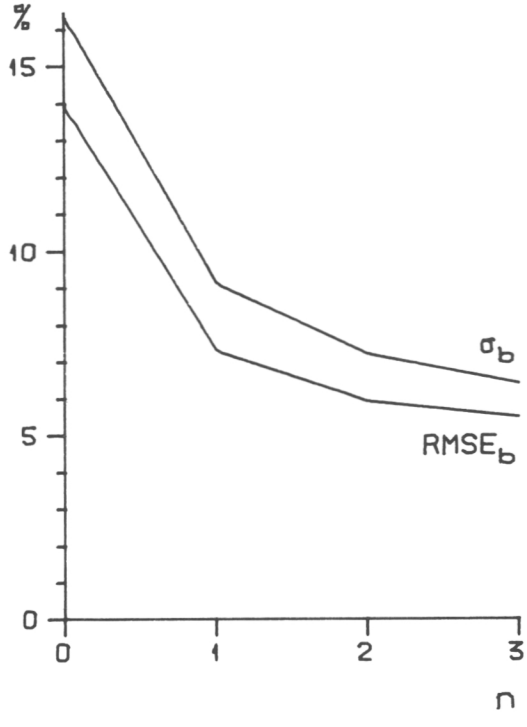

Fig. 26. Empirically estimated between-stand RMSE ($RMSE_b$) of the volume estimates compared with the theoretically derived between-stand standard error (σ_b) as a function of the number of calibrating trees. Data of Laasasenaho, overall model; only stands having at least three trees have been used.

Table 11. Theoretically derived between-stand, within-stand and total standard deviations (σ_b, σ_w and σ_t) compared with empirically estimated between-stand RMSE ($RMSE_b$), within-stand and total standard deviations (s_w and s_t) for different combinations of measurements.

measured	model	$RMSE_b$	σ_b	s_w	σ_b	s_t	σ_t
$D_{1.3}$	overall	15.8	16.2	11.8	12.1	19.8	20.3
	region.	12.7	12.6	11.3	11.7	17.0	17.2
$D_{1.3}$, H	overall	3.9	4.0	6.3	5.1	7.4	6.5
	region.	3.4	3.7	6.2	5.0	7.1	6.2
$D_{1.3}$, H, D_6	overall	1.7	2.8	3.3	3.8	3.7	4.7
	region.	1.7	2.7	3.2	3.8	3.6	4.6

differences between measurement combinations and between the overall and regionalized models.

Empirical and theoretical between-stand standard errors of the volume estimates for the tally trees are compared in Fig. 26 using the data of Laasasenaho. Theoretical standard error describes well the rate of calibration. As noted in section 65, the subdata used to test the calibration deviate from the whole data. Hence the empirical between-stand RMSE is on a lower level than the theoretical between-stand standard error.

74. Timber assortment problems

In timber assortment problems the interesting quantities (e.g., the saw-log volume) are discontinuous functions of the stem dimensions. The estimates of assortment volumes can be heavily biased, if we just use the predicted stem curves to estimate volumes without taking the prediction errors into account. Timber assortment problems are worth a separate study; in this section the applicability of the stem form model is only demonstrated.

A key question in timber assortment problems is whether the stem dimensions meet certain minimum requirements. From a statistical point of view, we should be able to compute joint and conditional probabilities that diameters at given heights fall within a specified range. Assuming that the random effects are normally distributed, the probabilities can be estimated using the variances and covariances of the prediction errors.

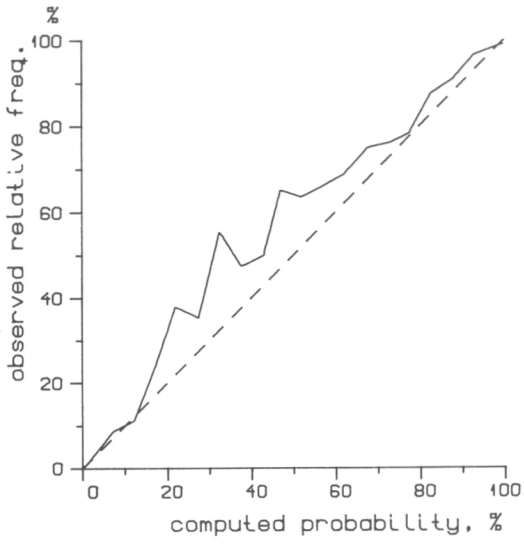

Fig. 27. Observed relative frequency of trees with $D_6>17$ as a function of the computed probability. Mean values within 5 %-intervals of the computed probability.

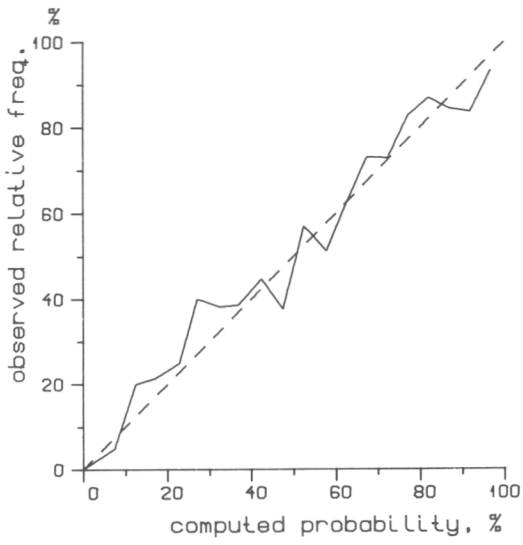

Fig. 28. Observed relative frequency of trees with $H>15$ and $D_6>17$ as a function of the computed probability. Mean values within 5 %-intervals of the computed probability.

In the first example, it is assumed that $D_{1.3}$ is known and we seek the probability that $D_6>17$ cm. This probability was computed for each tree as follows. The logarithmic stem curve was first predicted for the knot angles, and the variances of the prediction errors were computed. The nearest knot points (in the arithmetic scale) to the height of 6 m were next determined. Then the percentage points of the predicted diameters (defined in the logarithmic scale) of these angles were changed stepwise until the interpolated diameter (in the arithmetic scale) at 6 m was 17 cm. For simplicity, linear interpolation was used. The observed relative frequency is shown in Fig. 27 as function of the computed probability. Owing to the effect of the relative size, the computed probability is not a function of $D_{1.3}$ only. The correspondence is quite satisfactory; the slight underestimation results, at least partly, from the linear interpolation used.

If we are interested in two or more dimensions simultaneously, then joint (and conditional) probabilities are required. In the stem form model these could be obtained using covariances of the prediction errors. Because these covariances are not used elsewhere in this study, they were not computed, even though theoretically their derivation is similar to that for variances. However, joint probabilities can be also computed indirectly, as indicated in the following.

In the second example, $D_{1.3}$ was again assumed to be measured, and the joint probability for $H>15$ m and $D_6>17$ cm was estimated. First, the logarithmic stem curve was predicted, and the variances of the prediction errors were computed. Then the (logarithmic) height was changed stepwise along different percentage points. Each height greater than 15 m was used as a measurement together with the measured $D_{1.3}$, and the stem curve was predicted. Then the probability for $D_6>17$ cm was computed as in the first example. The joint probability of this elementary case is the probability of the corresponding height multiplied by the probability that $D_6>17$ cm. The total probability is then the sum of the elementary probabilities. The correspondence between the computed probability and observed relative frequency was good (Fig. 28).

75. Optimization of measurements

The stem form model can be applied to problems of stem form when any measurements of the stem are available. The stem form model was tested previously in different situations where standard measurements are used. In this section the stem form model is used to analyze different measurement strategies theoretically. The purpose of this section is to demonstrate how the stem form model can be easily applied to preliminary theoretical calculations.

Assume that we are measuring a diameter of a tree in a given stand. What is the optimal relative height for the measurement? According to the stem form model estimation errors depend mainly on the measurement angle. Using the average stem curve the angles can be expressed in approximative relative heights, which are better for descriptive purposes.

The significance of the relative measurement height of a single measured diameter was studied as follows. First $D_{1.3}$=20 cm was used as the only measurement for a tree. Then the logarithmically unbiased stem curve was predicted, and the points at the knot angles were used as possible measurements for the same tree. RMSE of the estimated volume was then calculated from the RMSE of the size parameter as described in section 53. There seems to be an optimum at approximately 40 % of the total height (Fig. 29). The reason for this optimum height can be seen from Fig. 7, which shows the effects of the principal components. The diameter at approximately 40 % of the height is nearly independent of the most important thickness component of the variation in stem form; thus it provides the most precise estimate of the stem size. Note that Laasasenaho (1982) found the 50 %-height to be slightly better. When the measurement costs are taken into account, the 40 %-height may be too high in most practical measurement situations. When both the measurement cost and the accuracy of the volume estimates are considered, however, a height greater than 1.3 m might be reasonable as the principal measurement height (as suggested, e.g., by Cajanus 1911).

Next we determine the optimal point for measuring diameter if the height is measured. The calculations were made in the same way as for a single measurement. The

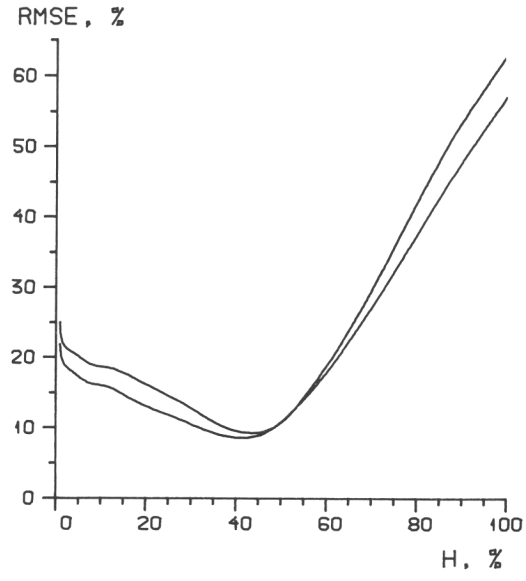

Fig. 29. RMSE of the volume estimate as a function of the relative height of the measured diameter. The relative height has been computed using an average tree. The upper curve is for the overall model and the lower curve for region 5 in the regionalized model.

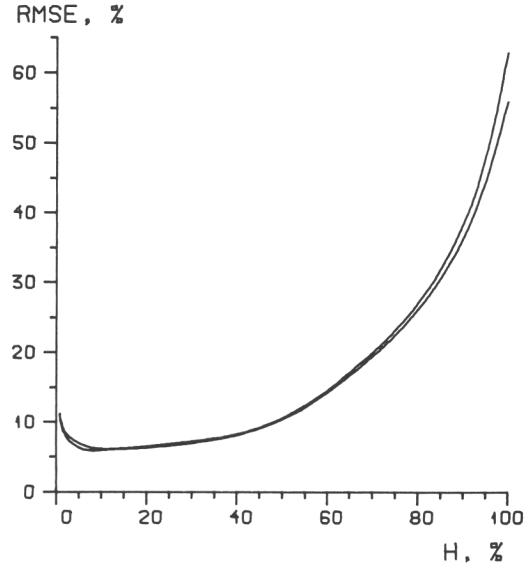

Fig. 30. RMSE of volume estimate as a function of the relative height of the measured diameter measured in addition to the height. The upper curve is for the overall model and the lower curve for region 5 in the regionalized model.

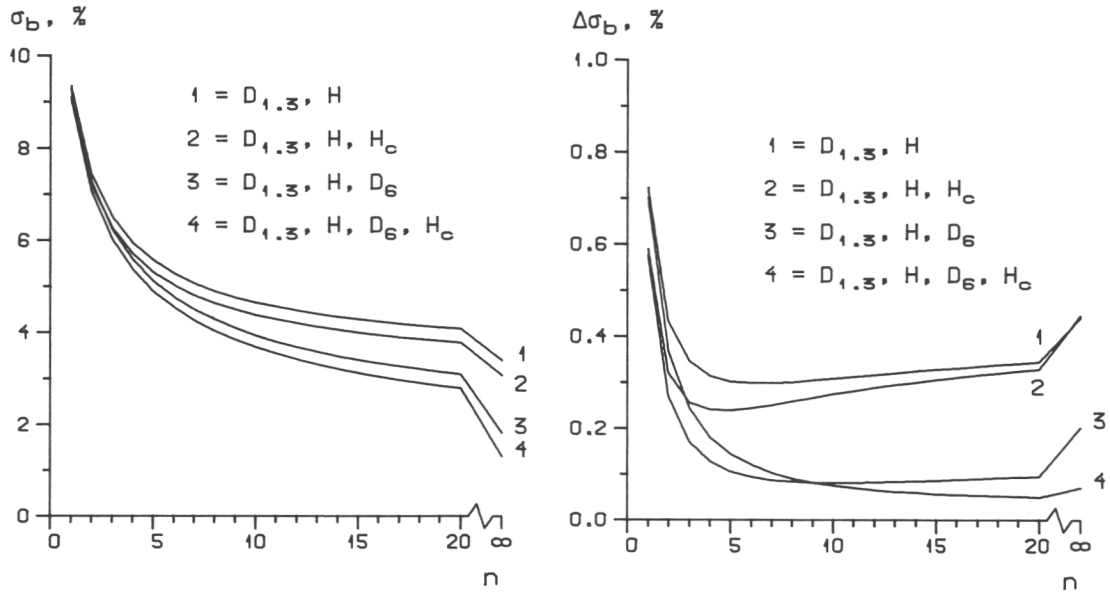

Fig. 31. σ_b as a function of the number of calibrating trees for different calibrating measurements (a) and the difference between σ_b of the overall model and σ_b of the regionalized model (b).

'measured' height was taken from the expected stem curve for $D_{1.3}$=20 cm: H is 14.1 m for the overall model and 15.2 m for region 5 in the regionalized model. Now the optimal relative height for the diameter measurement is approximately 10 % (Fig. 30). RMSE changes so slowly that any height between 5 % and 40 % serves almost as well. Laasasenaho (1982) found the 30 %-height to be optimal. If the height of the tree and one diameter are measured, then a measurement height of 1.3 m is reasonable in most practical measurement situations.

Calibration of the stem curve can also be studied theoretically. We can generate as many calibrating trees as we want and then survey the estimated between-stand standard error of the volume estimate of a tally tree. If the measurements are made at fixed absolute heights, then the theoretical results depend on the properties (mainly size) of the trees used in the analysis. For simplicity, the computations were made using $D_{1.3}$=20 cm for all trees and, in addition, H=14 m, D_6=15 cm and H_c=6 m for the calibrating trees. These measurements correspond roughly to the average stem form. The measurement combinations for the calibrating trees were ($D_{1.3}$, H), ($D_{1.3}$, H, H_c), ($D_{1.3}$, H, D_6) and ($D_{1.3}$, H, D_6, H_c). The theoretical standard deviation of the between-stand error, σ_b, is presented in Fig. 31a as a function of the number of calibrating trees; the difference between the overall and regionalized models is shown in Fig. 31b.

As expected, σ_b is greatest for combination ($D_{1.3}$, H) and smallest for combination ($D_{1.3}$, H, D_6, H_c). With a small number of calibrating trees ($n \leq 2$ in the overall model and $n \leq 5$ in the regionalized model) σ_b of combination ($D_{1.3}$, H, H_c) is smaller than σ_b of combination ($D_{1.3}$, H, D_6). Because the within-stand variation of H_c is small compared to the between-stand variation (see Table 2), measurements of H_c will rapidly provide information about the stand-effects of the taper form. D_6 contains more information about the stand stem form than H_c does, but because of the relatively large within-stand variation, we need more measurements to obtain this information.

The difference between the overall and the regionalized model is significant for calibrating measurement combinations ($D_{1.3}$, H) and ($D_{1.3}$, H, H_c) even if the number of calibrating trees is large. This agrees with the results of section 63. Only the measurement combination ($D_{1.3}$, H, D_6, H_c) seems to reveal practically all the regional differences.

If the calibrating measurement combinations are to be selected optimally, the measurement costs must also be taken into account. Let us use the same relative measurement costs as Kilkki (1983), i.e., the relative costs are:

combination	cost
$D_{1.3}$, H	1.0
$D_{1.3}$, H, H_c	1.1
$D_{1.3}$, H, D_6	1.28
$D_{1.3}$, H, D_6, H_c	1.38

In Fig. 32 σ_b is presented as a function of the measurement cost. For small measurement costs the combination ($D_{1.3}$, H, H_c) is optimal, and for higher costs the combination ($D_{1.3}$, H, D_6, H_c) is optimal. Thus the combination ($D_{1.3}$, H, D_6), which is commonly used in Finland, is not optimal for any cost level. These results agree with those of Kilkki (1983), which were derived using a different approach. If fixed overhead costs are taken into account, it is not as advantageous to measure additional dimensions (e.g., D_6) as indicated in Fig. 32.

If, in the optimization of the measurement height or the calibrating measurement combination, trees are assumed to have different dimensions than above, the estimated error variances will also change. The above conclusions seemed, however, to be qualitatively valid for different stem types tested.

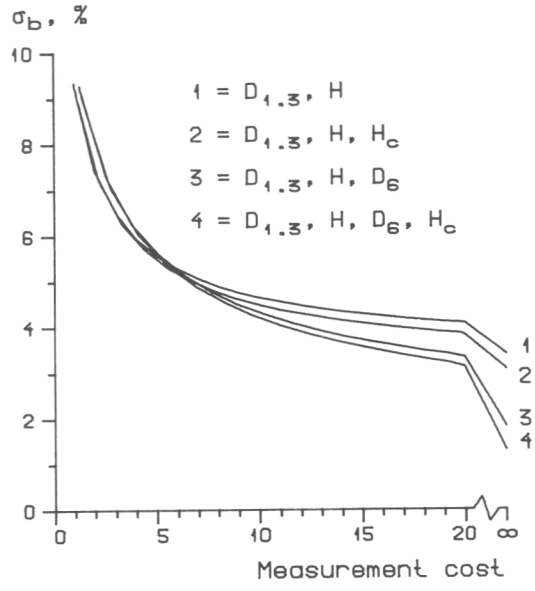

Fig. 32. σ_b as a function of the measurement cost for different calibrating measurements; the overall model.

Note that the results attained in the polar coordinate system can be interpreted only approximately in the height-diameter coordinates. Definite recommendations can be made only after empirical tests designed for specific practical applications.

8. DISCUSSION

81. Generalization of the results

The model parameters were estimated using the comprehensive data of Laasasenaho (1982). Because those data were collected with a different modelling approach in mind, generalization of the results to the target population presents certain problems.

First, the measured trees were selected using a relascope, i.e., the sampling probabilities were proportional to the basal area of the trees. In the analysis, tree size is treated as a fixed variable and the random effects are assumed to be independent of the size of the tree. This would be the case, if the probability of sampling were a function of size or were independent of any tree characteristics. But as the sampling probabilities are proportional to the basal arrea, it appears that, e.g., the thicker one of two nearby trees of equal size has had a greater sampling probability than the thinner one. Thus, according to the estimated model, the expected stem forms are too thick. Because basal area and size are highly correlated, however, the bias is probably quite small. Only thicker trees in a given stand have had too great a probability of selection; thicker stands have not had greater selection probabilities than the thinner stands. Thus only the within-stand random effects intervene in the selection of sample trees. Recall that 'thickness' means relative thickness after adjusting for size. When the model is applied for trees selected by relascope, the results should again be unbiased.

Secondly, our interpretation of 'stand' is not in accordance with the common view. Each sample plot in the data was assumed to represent a stand. A stand is generally understood to mean a larger area in which the average correlations between trees are not as large as in the 'stands' of this study. With larger stands the variances of the within-stand effects would be greater than estimated in this study; the correlations would change much less. In applications where the correlation structures are more important than the variances of between-stand and within-stand effects, this deficiency is not very serious. Test results with the independent test data of Kilkki and Varmola and of Pekkonen and Laasasenaho support this view.

The small number of trees per stand in the data also causes another problem in the estimation of the model parameters. The average size of the trees in the stand, $\bar{s}$, is needed in the model to determine the expected value of the random stand effects. The relative size of a tree, $s_i - \bar{s}$, also explains how the competitive status of a tree affects the stem form. In the model $\bar{s}$ is treated as a fixed variable. Because of the small number of trees per stand, however, $\bar{s}$ measures quite roughly the true average size in the measured stands. On the other hand, it may also be reasonable to determine the competitive status of a tree using only the nearby trees.

In applications $\bar{s}$ is computed as the mean of the preliminary size estimates. The estimated $\bar{s}$ is in accordance with $\bar{s}$ used in the analysis, if it is computed as a weighted average of the sizes, the weights being proportional to the basal area. If the trees have been selected with a relascope, the average can again be estimated by the ordinary arithmetic mean.

82. Development of the model

There are several possible ways to extend and refine the model. During the course of the study many slight modifications were already tested. For instance, the following alternatives did not work better than the methods described in this study:

(i) Parameters were interpolated by natural splines or by quasi-Hermite piecewise polynomials.

(ii) Average size $\bar{s}$ was estimated in the applications as a separate parameter.

(iii) Preliminary estimates of size were used to estimate only $\bar{s}$ and not $s_i - \bar{s}$.

Preliminary calculations with the data of Kilkki and Varmola and the comparisons with Pekkonen's calibration system in the data of Pekkonen and Laasasenaho indicated that the dependence of stem form on size may vary slightly from stand to stand. Thus it might be reasonable to include a term $b_k(u)s_{ki}$ into the explanatory variables of the logarithmic diameter d(u), where b_k would be a random parameter that changes from stand to stand. In the analysis this refinement could be handled by standard methods. In the applications, however, this new term would be a product of two unknown parameters, which could not be estimated simultaneously by standard methods. An iterative estimation procedure might still work. In the first round the term $b_k(u)s_{ki}$ could be neglected or s_{ki} could be replaced by the preliminary estimate already used in the estimation. In the second round s_{ki} could be replaced by the estimate obtained in the first round. Owing to the identification problems involved, any work along such lines should be very careful.

According to the model presented, the covariance matrix of the within-stand effects is the same for all stands. This is not exactly true, because stands are not equally homogenous. For instance, stem forms in a stand are probably more alike after a selective thinning than before. Estimation of a different within-stand covariance matrix for each stand would be possible only using a covariance function with a few parameters. The simplest possibility is to multiply a constant covariance matrix by a variable scale factor. In this case only the weighting of between-stand and within-stand covariance matrices would change from stand to stand. Separate covariance matrices for some subdomains (e.g., for climatic regions) could easily be estimated using any sufficiently representative data.

The model was derived for fixed angles in the polar coordinate system. In applications the measurements can be at any angles, and the needed parameters are interpolated by cubic splines. One-dimensional splines are used to interpolate fixed parameters, characteristic vectors and covariances between measurement angles and the knot angles; two-dimensional splines are used to interpolate covariances between measurement angles. It may be possible to replace the splines with ordinary functions, but mathematically these functions are probably as unsophisticated as the splines used.

The analysis of stem form variation was based on variables measured from the trees themselves. Crown height was the only exogenous variable used in the analysis (in addition to the 'measurement' of climatic region and stand index). All other variables, except stem dimensions, are here called 'exogenous'. In future development of the model it should be possible to incorporate other exogenous stand variables or tree variables into the model. There are several ways to do this.

Exogenous tree variables could be treated in the same way as the crown height. For instance, crown depth (height — crown height) might be a reasonable variable to include. In logarithms, the crown depth cannot be expressed as a linear combination of height and crown height, thus it can bring to a linear model information not present in height and crown height alone. Variables added in this way do not change the previous part of the model.

Any explanatory stand variables or tree variables could also be added to the equations describing logarithmic diameters at different angles. For instance, adding the basal area of the stand would clearly improve the model. But then the estimation of the model parameters should be repeated. In different applications different sets of exogenous variables are known, and there should then be different models for different situations. This would, however, contradict the goal of finding a single unifying model applicable in many different situations. Perhaps the principal components of the random stand and tree effects can provide a flexible way of formulating and utilizing the information about new variables. For instance, it may be possible to formulate simple regression equations for c_1, the first principal component of the stand effects. Then the estimation procedure can easily be modified accordingly.

We have seen that, using a single height measurement, the stem curve can already be calibrated quite accurately for a given stand. Hence the usefulness of exogenous stand information is questionable, if we are taking any measurements in the stand. The situation is different if the estimation is based

on general information available, i.e., without field measurements.

Growth modelling may become an appealing area in which this approach can be applied. With the stem form model the change in size can be separated from the change in stem form. The simplest way to predict the path of development of a stem curve is to assume that the random stand and tree effects remain constant, and that only the size is increasing. In that case the development of a stem curve can be predicted by predicting the growth in size. More complex statistical analysis is needed if development of the random stand and tree effects over time is also described. In the growth modelling it should be taken into account the fact that in this study the model was developed for cross-sectional data, i.e., for trees at a given point in time. Because the natural deaths or cuttings of trees are correlated with the stem forms of trees, the average stem form development of individual trees may not follow the expected stem curves of this study.

The similarity of trees in the same stand is formulated in this study using random stand effects. This similarity can also be described without stand effects by assuming that within a stand the tree effects are correlated. Growth modelling may be easier with correlated tree effects than with random stand effects. Stem forms could also be described in an intuitively appealing way with principal components, if each tree could be described using a single set of principal components.

The polar coordinate system was used originally by Sloboda (1977a) to provide good projections of growth. He also studied curvilinear coordinate systems in order to express the change in stem form as size increases. With the approach used in the present study the dependence of stem form on size can be expressed more simply.

Extension of the model to other tree species would probably be quite straightforward. The approach presented may also be applicable in studies of the shapes of other organisms. For instance, in the stem form model the independence of shape and size, a topic widely discussed in the literature, is just a special case. When the dimensions are linked to each other through the size, an artificial variable, and through a simple covariance structure, we need not make a complicated nonlinear system for the interdependence of different dimensions.

9. SUMMARY

In this study a general model for variation in the stem form of Scots pine is presented. The model gives a compact description of how stem form varies as a multidimensional object. The dimensions are determined in polar coordinates.

Most of the apparent variation in stem dimensions can be attributed to differences in size. The size is first defined as a weighted mean of logarithmic dimensions. Thereafter the logarithmic dimensions are described by a simple linear model in which size is treated as an ordinary fixed variable. The model contains a fixed part, which describes how the expected (average) stem form depends on the size and relative size of the tree.

Using random stand and tree effects, the random variation of the stem dimensions is partitioned into variation between stands and variation within stands. The variances and covariances of the random effects and the fixed parameters of the model were estimated using standard techniques for mixed linear models. The principal components of the between-stand and within-stand covariance matrices were used to give a more economical description of the main directions of the random variation of the stem form.

In applications, the roles of parameters and variables are changed. After some approximative derivations, the model can again be presented in the standard form. The sizes of the trees and the random stand effects are the parameters to be estimated. Random stand effects are estimated using the first few principal components of the between-stand covariance matrix. The model can be applied when any stem dimensions are measured.

With usual measurements the model is as good as the regression equations of Laasasenaho (1982). The model can be used to calibrate the stem curve for a given stand; the calibrated stem curve can then be applied for tally trees, i.e. trees with one measured dimension. The stem curve is calibrated by estimating the random stand effects. By measuring the height and diameter at breast height of a single tree, the between-stand variance of the volume estimates of tally trees can already be reduced by 70 %. With a small number of calibrating measurements, the model gives better results than the calibration system of Pekkonen (1982).

The model also estimates the error variances of the predicted stem curves and volumes. It is demonstrated how error variances can be applied in timber assortment problems. Measurement errors can be corrected to some extent by incorporating the variances of the measurement errors into the model.

The model can be used to study different measurement strategies. If only one stem dimension is measured, the optimal measurement height seems to be above the commonly used 1.3 m. Theoretical analysis indicates that the calibrating measurements currently used in Finland are not optimal.

The stem form model is based on the standard theory of mixed linear models. Apparently, the most compact prediction formulas for a mixed linear multivariate model are not available elsewhere, and are therefore derived in this study.

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SELOSTE

Männyn runkomuodon vaihtelun analysointi ja ennustaminen lineaaristen sekamallien avulla

Analyysi

Tutkimuksessa esitetään yleinen männyn runkomuodon vaihtelun malli. Mallissa runkomuoto kuvataan läpimittojen moniulotteisena vektorina. Läpimitat ilmaistaan napakoordinaatistossa. Puun koko määritellään logaritmisten läpimittojen painotettuna keskiarvona. Logaritmiset läpimitat kuvataan yksinkertaisella lineaarisella mallilla, missä puun koko ja suhteellinen koko ovat kiinteitä selittäjiä. Satunnaisten metsikkö- ja puutekijöiden avulla runkomuodon satunnaisvaihtelu jaetaan metsiköiden väliseen ja metsikön sisäiseen vaihteluun. Kiinteät parametrit sekä satunnaisvaikutusten varianssit ja kovarianssit estimoitiin käyttäen yleisiä lineaaristen sekamallien menetelmiä sekä Laasasenahon keräämää aineistoa.

Männyn keskimääräinen runkomuoto on keskikokoisilla puilla solakampi kuin pienillä tai isoilla puilla. Puut ovat sitä tukevampia mitä suurempia ne ovat verrattuna metsikön muihin puihin. Metsiköiden välisen ja metsikön sisäisen vaihtelun kovarianssimatriiseja analysoitiin pääkomponenttien avulla. Suurin osa sekä metsiköiden että yksittäisten puiden runkomuotojen satunnaisvaihtelusta on puun solakkuuden vaihtelua.

Noin puolet metsiköiden runkomuodon vaihtelusta liittyy ilmastovyöhykkeisiin. Mallin parametrit estimoitiin sekä koko aineiston avulla että kullekin ilmastovyöhykkeelle erikseen. Puut ovat keskimääräistä tukevampia Lapissa, Lounais-Suomen rannikolla ja kitu- ja joutomailla. Lapissa puun suhteellinen koko ei vaikuta runkomuotoon samalla tavalla kuin muualla Suomessa.

Mallin soveltaminen

Mallia sovellettaessa puun koko tulkitaan satunnaiseksi tai kiinteäksi parametriksi, ja analyysivaiheen parametrit ovat muuttujia. Malli kalibroidaan metsikkökohtaisesti estimoimalla satunnaiset metsikkötekijät pääkomponenttien avulla. Mallia voidaan soveltaa olipa metsikön puista mitattu mitä tahansa dimensioita (läpimitat, pituus, latvusraja). Splinien avulla analyysivaiheen parametrit interpoloidaan käytetyille mittauskorkeuksille.

Mittausten avulla ennustetaan logaritmiset läpimitat analyysivaiheessa käytetyille peruskulmille. Kun logaritmiset läpimitat muunnetaan aritmeettiseen mittausasteikkoon, käytetään läpimitan ja sen ennustimen varianssista riippuvaa korjauskerrointa harhattoman ennustimen saamiseksi. Tämän korjauskertoimen merkitys on kuitenkin pieni, jos puun koko tulkitaan kiinteäksi parametriksi. Koko runkokäyrä kuvataan splinillä, joka määritellään peruskulmissa ennustettujen läpimittojen avulla.

Ennustettu runkokäyrä kulkee mitattujen pisteiden kautta, jos mittausten tiedetään olevan virheettömiä. Jos mittausvirheiden varianssi tunnetaan, mittausvirheitä voidaan mallin avulla korjata. Mallin avulla saadaan myös estimaatit ennustettujen läpimittojen virhevarianssille. Rungon tai sen osan tilavuus estimoidaan integroimalla ennustettua runkokäyrää. Tilavuutta estimoitaessa käytetään eri harhattomuuskorjausta kuin läpimittoja ennustettaessa. Tilavuusestimaatin metsiköiden välinen ja metsikön sisäinen virhevarianssi voidaan estimoida käyttäen kokoparametrin vastaavia virhevariansseja.

Yleisillä mittauskombinaatioilla malli antaa yhtä hyviä tuloksia kuin normaalit regressiomallit. Lukupuiden tilavuusestimaattien metsikkövirhe pienenee jo 70 prosentilla, kun yhdestä koepuusta on mitattu pituus ja rinnankorkeusläpimitta. Ennustettujen läpimittojen virhevarianssien avulla voidaan estimoida harhattomasti eri puutavaralajien määriä.

Mallin avulla voidaan tutkia teoreettisesti mittausstrategioita. Noin 40 %:n korkeudelta mitattu läpimitta antaa tarkimman tilavuusestimaatin, jos puusta mitataan vain yksi dimensio. Runkokäyrän metsikkökohtaista kalibrointia varten kannattaisi koepuista mitata vain pituus ja rinnankorkeusläpimitta, jos koepuumittauksia tehdään vähän. Jos koepuumittauksia tehdään paljon kannattaisi pituuden, rinnankorkeusläpimitan ja yläläpimitan lisäksi mitata myös latvusraja. Mittausvirheiden merkitystä voidaan analysoida mallin avulla.

Tutkimuksessa sovelletaan yleistä lineaaristen sekamallien teoriaa. Yleinen lineaarinen ennustin ja sen virhevarianssi johdetaan sekamalleissa helpommin soveltuvaan muotoon.

LAPPI, J. 1986. Mixed linear models for analyzing and predicting stem form variation of Scots pine. Seloste: Männyn runkomuodon analysointi ja ennustaminen lineaaristen sekamallien avulla. Commun. Inst. For. Fenn. 134: 1—69.

APPENDIX

A. MIXED LINEAR MODELS

A.1. Model

Results for the mixed linear models that are used in the study are presented here. The main references are Searle (1971) and Harville (1977). Prediction formulas derived in section A.4 were not available elsewhere; therefore, their detailed derivation is presented.

Most linear statistical models are special cases of the following general model:

$$\mathbf{y} = \mathbf{Xa} + \mathbf{Zb} + \mathbf{e}, \qquad \text{(A.1.1)}$$

where $\mathbf{y}$ is an N×1 vector of random variables; $\mathbf{X}$ and $\mathbf{Z}$ are matrices of regressors with dimensions N×q and N×p, respectively; $\mathbf{a}$ is a q×1 vector of fixed effects (parameters); $\mathbf{b}$ is a p×1 vector of unobservable random effects (parameters); and $\mathbf{e}$ is an N×1 vector of random errors. It is assumed that $E(\mathbf{b}) = \mathbf{0}$, $E(\mathbf{e}) = \mathbf{0}$, and $\text{cov}(\mathbf{b},\mathbf{e}') = \mathbf{0}$. Let $\mathbf{D}=\text{var}(\mathbf{b})$, $\mathbf{R}=\text{var}(\mathbf{e})$, and $\mathbf{V}=\text{var}(\mathbf{y}) = \mathbf{R} + \mathbf{ZDZ}'$. To avoid unnecessary complications, we assume that $\mathbf{X}$ and $\mathbf{Z}$ are of full column rank; later all models are also defined in such a way that we can work with inverses of matrices instead of generalized inverses.

All usual regression and analysis of variance models are special cases of this model, as well as multivariate, time-series and factor analysis models. Usually $\mathbf{D}$ and $\mathbf{R}$ have a special structure (e.g. diagonal) so that there are fewer unknown parameters than elements in $\mathbf{D}$ and $\mathbf{R}$.

In the statistical literature, determination of $\mathbf{b}$ from the data is termed either 'estimation' or 'prediction'. Here also the random parameter vector $\mathbf{b}$ is 'estimated'; the unknown y's are 'predicted'. Note that usually, but not in our case, the estimation of $\mathbf{b}$ and the prediction of y's coincide.

The diagonal elements of $\mathbf{D}$ and $\mathbf{R}$ are called variance components; the nondiagonal elements are covariance components. There are several methods of estimating $\mathbf{D}$ and $\mathbf{R}$. If $\mathbf{D}$ and $\mathbf{R}$ are known, the estimation of $\mathbf{a}$ and $\mathbf{b}$ is straightforward. Let us consider this case first.

A.2. Estimation of fixed and random parameters

Theoretically, the estimation of the fixed parameter vector $\mathbf{a}$ in (A.1.1) is simple. The generalized least squares (GLS) estimator is the best (minimum variance) linear unbiased estimator. As $\text{var}(\mathbf{y})=\mathbf{V}$, the GLS estimate of $\mathbf{a}$ is:

$$\hat{\mathbf{a}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y} \qquad \text{(A.2.1)}$$

As the matrix $\mathbf{V}$ is of order N, and $\mathbf{V}$ is not generally diagonal, even if $\mathbf{D}$ and $\mathbf{R}$ are, the direct inversion of $\mathbf{V}$ is, in most cases, impossible. However, the best linear estimates of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ can be obtained simultaneously from:

$$\begin{bmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{D}^{-1} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{y} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{y} \end{bmatrix}. \qquad \text{(A.2.2)}$$

Let

$$\mathbf{H} = \begin{bmatrix} \mathbf{X}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{X}'\mathbf{R}^{-1}\mathbf{Z} \\ \mathbf{Z}'\mathbf{R}^{-1}\mathbf{X} & \mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{D}^{-1} \end{bmatrix}. \qquad \text{(A.2.3)}$$

Then $\mathbf{H}^{-1}$ is also the covariance matrix of the estimation errors, i.e., its submatrices are $\text{var}(\hat{\mathbf{a}})$, $\text{cov}(\hat{\mathbf{a}},\hat{\mathbf{b}}'-\mathbf{b}')$ and $\text{var}(\hat{\mathbf{b}}-\mathbf{b})$. If $\mathbf{D}$ is singular, $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ can be solved from equations given by Harville (1977). Assuming normality, $\hat{\mathbf{b}}$ is the best unbiased and not only the best linear estimator.

If both $\mathbf{a}$ and $\mathbf{b}$ are fixed, they should be estimated using (A.2.2) after $\mathbf{D}^{-1}$ is dropped from the left side. $\mathbf{D}^{-1}$ brings into the model the prior information about the distribution of the $\mathbf{b}$-effects.

Vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ can be solved directly from (A.2.2). Another possibility is to apply

the following representation for $\mathbf{V}^{-1}$:

$$\mathbf{V}^{-1} = \mathbf{R}^{-1} - \mathbf{R}^{-1}\mathbf{Z}(\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z}+\mathbf{D}^{-1})^{-1}\mathbf{Z}'\mathbf{R}^{-1}. \quad \text{(A.2.4)}$$

Thereafter $\hat{\mathbf{a}}$ can be computed using (A.2.1) and $\hat{\mathbf{b}}$ can be computed as:

$$\hat{\mathbf{b}} = (\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z} + \mathbf{D}^{-1})^{-1}\mathbf{Z}'\mathbf{R}^{-1}(\mathbf{y}-\mathbf{X}\hat{\mathbf{a}}). \quad \text{(A.2.5)}$$

Conversely, we can first compute $\hat{\mathbf{b}}$ and thereafter $\hat{\mathbf{a}}$:

$$\hat{\mathbf{b}} = (\mathbf{Z}'\mathbf{S}\mathbf{Z} + \mathbf{D}^{-1})^{-1}\mathbf{Z}'\mathbf{S}\mathbf{y}, \text{ where} \quad \text{(A.2.6)}$$

$$\mathbf{S} = \mathbf{R}^{-1} - \mathbf{R}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{R}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{R}^{-1}, \text{ and then} \quad \text{(A.2.7)}$$

$$\hat{\mathbf{a}} = (\mathbf{X}'\mathbf{R}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{R}^{-1}(\mathbf{y}-\mathbf{Z}\hat{\mathbf{b}}). \quad \text{(A.2.8)}$$

Assuming normality, the presented estimates of **a** and **b** are also Bayes estimates. In the Bayesian approach the parameter vector **a** is also considered to be random but with very large variance.

Variances of the elements of $\hat{\mathbf{a}}$ can also be partitioned by utilizing the fact that any element $\hat{a}_j$ of $\hat{\mathbf{a}}$ is linear combination of the observed y's, i.e., $\hat{a}_j = \mathbf{c}'\mathbf{y}$ for some vector **c**. Hence

$$\hat{a}_j = \mathbf{c}'\mathbf{y} = \mathbf{c}'(\mathbf{X}\mathbf{a} + \mathbf{Z}\mathbf{b} + \mathbf{e}), \text{ and}$$

$$\text{var}(\hat{a}_j) = \mathbf{c}'\mathbf{Z}\mathbf{D}\mathbf{Z}'\mathbf{c} + \mathbf{c}'\mathbf{R}\mathbf{c}. \quad \text{(A.2.9)}$$

If **R** is block diagonal, the decomposition can be developed further. Let **c** be partitioned into subvectors $\mathbf{c}_i$ corresponding to the blocks $\mathbf{R}_i$ of **R**. Then

$$\text{var}(\hat{a}_j) = \mathbf{c}'\mathbf{Z}\mathbf{D}\mathbf{Z}'\mathbf{c} + \sum_i \mathbf{c}_i'\mathbf{R}_i\mathbf{c}_i. \quad \text{(A.2.10)}$$

This decomposition is used later on, when estimation errors are partitioned into between-stand and within-stand errors. Similarly, the variances of $\hat{\mathbf{b}}$ or $\hat{\mathbf{b}}$—**b** can be decomposed into components.

A.3. Estimation of variance and covariance components

Estimation of the variance and covariance components from unbalanced data has attracted considerable attention in recent years. Nowadays there are several different methods available, e.g., the three methods of Henderson, maximum likelihood (ML), restricted ML (REML), minimum variance quadratic unbiased estimator (MIVQUE), and minimum norm quadratic unbiased estimator (MINQUE). For ML and REML the reader is refered to Harville (1977) or Dempster et al. (1981), and for MIVQUE and MINQUE to Searle (1979).

In this study the variance and covariance components were estimated using the fitting constants method (Henderson's method 3; see Searle 1971). This technique can be used in mixed models and was easy to implement in the present case. During the course of the study the maximum likelihood method was also used; the EM-algorithm as presented in Dempster et al. (1981) was generalized to the multivariate situation. However, using this more complicated method, no real gains were obtained.

In order to describe the fitting constants method, we start from the model

$$\mathbf{y} = \mathbf{X}\mathbf{a} + \mathbf{Z}\mathbf{b} + \mathbf{e}, \quad \text{(A.3.1)}$$

where **a** can now contain both fixed and random parameters, **b** contains only random effects, and **e** is the random error, uncorrelated with other random effects, and $E(\mathbf{e}) = \mathbf{0}$ and $\text{var}(\mathbf{e}) = \sigma_e^2\mathbf{I}$.

First, we fit the following reduced model by ordinary least squares:

$$\mathbf{y} = \mathbf{X}\mathbf{a} + \mathbf{e}. \quad \text{(A.3.2)}$$

Let the resulting residual sum of squares be SSE(**a**). The next step is to fit the full model (A.3.1); let the resulting residual sum of squares be SSE(**a**,**b**). An unbiased estimate of σ_e^2 is

$$\hat{\sigma}_e^2 = \text{SSE}(\mathbf{a},\mathbf{b})/[\text{N-rank}(\mathbf{X}_*)], \text{ where} \quad \text{(A.3.3)}$$

$$\mathbf{X}_* = [\mathbf{X}\ \mathbf{Z}]. \quad \text{(A.3.4)}$$

The expected value of SSE(**a**)—SSE(**a**,**b**) is

$$E[\text{SSE}(\mathbf{a})-\text{SSE}(\mathbf{a},\mathbf{b})] = \quad \text{(A.3.5)}$$
$$\text{tr}\{\mathbf{Z}'[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{Z}\ E(\mathbf{b}\mathbf{b}')\}$$
$$+\sigma_e^2[\text{rank}(\mathbf{X}_*)-\text{rank}(\mathbf{X})].$$

Let k be the number of unknown variances (σ_e^2 and k—1 variances of random effects). Using (A.3.3) for the full model and

(A.3.5) for different partitioned models, we can form and solve a linear system of k equations and k unknowns. The resulting estimates are quadratic forms in the observations. Generally the fitting constants method is not uniquely determined: it is possible to form more equations than there are unknowns. In our special case the fitting constants method is, however, uniquely determined.

Because the stem form model is a multivariate model, the assumption $\text{var}(\mathbf{e})=\sigma_e^2\mathbf{I}$ does not hold. As indicated by Searle and Rounsaville (1974), the estimates of the covariance components can, however, be obtained simply by the fitting constants or by any similar method. As noted, the fitting constants method gives estimates of variance components which are quadratic forms in the observations. The corresponding estimates of covariance components are bilinear forms in the observations with the same coefficient matrix. It can be shown that the following well-known identities are also true for the covariance component estimates (and not only for the covariances):

$$\text{var}(x+y) = \text{var}(x) + \text{var}(y) + 2\text{cov}(x,y), \text{ or}$$

$$\text{cov}(x,y) = \frac{1}{2}[\text{var}(x+y) - \text{var}(x) - \text{var}(y)]. \quad \text{(A.3.6)}$$

Therefore, the fitting constants estimates for covariance components can be obtained by estimating the variance components for all pairwise sums of variates.

A.4. Prediction of new observations

When applying the stem form model, we predict those dimensions which have not been measured. Here the prediction problem is discussed in general terms. The predictor formula of Goldberger (1962) is developed further for mixed linear models. Moreover, variances of the predictor and the prediction error will be derived. Let us assume the following model for the observed values of y:

$$\mathbf{y} = \mathbf{Xa} + \mathbf{u}, \quad \text{(A.4.1)}$$
$$E(\mathbf{u}) = \mathbf{0}, \quad \text{(A.4.2)}$$
$$\text{var}(\mathbf{y}) = \text{var}(\mathbf{u}) = \mathbf{V}, \quad \text{(A.4.3)}$$

where $\mathbf{y}$ is an N×1 vector, $\mathbf{X}$ is a fixed N×q matrix, $\mathbf{a}$ is a q×1 parameter vector, and $\mathbf{u}$ is an N×1 vector of random errors. Note that the general mixed model (A.1.1) is a special case where $\mathbf{u} = \mathbf{Zb} + \mathbf{e}$. Let $\hat{\mathbf{a}}$ denote the GLS estimate of $\mathbf{a}$:

$$\hat{\mathbf{a}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}. \quad \text{(A.4.4)}$$

The problem is to predict a single value of regressand y_0 given the vector of regressors $\mathbf{x}_0$. We can write

$$y_0 = \mathbf{x}_0'\mathbf{a} + u_0, \quad \text{(A.4.5)}$$

where u_0 is the random error for which $E(u_0)=0$.

Denote that

$$\mathbf{c} = \text{cov}(\mathbf{u},u_0) = [\text{cov}(u_1,u_0), \ldots, \text{cov}(u_n,u_0)]'. \quad \text{(A.4.6)}$$

Then according to Goldberger (1962), the best linear unbiased predictor of y_0, $\hat{y}_0$ is

$$\hat{y}_0 = \mathbf{x}_0'\hat{\mathbf{a}} + \mathbf{c}'\mathbf{V}^{-1}(\mathbf{y}-\mathbf{X}\hat{\mathbf{a}}). \quad \text{(A.4.7)}$$

Assuming normality, if $\hat{\mathbf{a}}=\mathbf{a}$, then $\mathbf{y}-\mathbf{X}\hat{\mathbf{a}}=\mathbf{u}$ and the predictor $\hat{y}_0$ would be the conditional expectation of y_0 given $\mathbf{y}$.

For a mixed model $\mathbf{y}=\mathbf{Xa}+\mathbf{Zb}+\mathbf{e}$ the formula (A.4.7) can be developed further by substituting the expression (A.2.4) for $\mathbf{V}^{-1}$:

$$\hat{y}_0 = \mathbf{x}_0'\hat{\mathbf{a}} + \mathbf{c}'\mathbf{V}^{-1}(\mathbf{y}-\mathbf{X}\hat{\mathbf{a}})$$

$$= \mathbf{x}_0'\hat{\mathbf{a}} + \mathbf{c}'[\mathbf{R}^{-1}-\mathbf{R}^{-1}\mathbf{Z}(\mathbf{Z}'\mathbf{R}^{-1}\mathbf{Z}+\mathbf{D}^{-1})^{-1}\mathbf{Z}'\mathbf{R}^{-1}](\mathbf{y}-\mathbf{X}\hat{\mathbf{a}})$$

$$= \mathbf{x}_0'\hat{\mathbf{a}} + \mathbf{c}'\mathbf{R}^{-1}(\mathbf{y}-\mathbf{X}\hat{\mathbf{a}}-\mathbf{Z}\hat{\mathbf{b}}). \quad \text{(A.4.8)}$$

Equation (A.2.5) was used above to get $\hat{\mathbf{b}}$ into the formula.

This formula can be developed further by writing the random error u_0 in terms of the random effects $\mathbf{b}$:

$$u_0 = \mathbf{z}_0'\mathbf{b} + e_0, \quad \text{(A.4.9)}$$

where $\text{cov}(\mathbf{b},e_0) = \mathbf{0}$. Denote

$$\mathbf{w} = \text{cov}(\mathbf{e},e_0). \quad \text{(A.4.10)}$$

Thus $\mathbf{c} = \text{cov}(\mathbf{u},u_0) = \text{cov}(\mathbf{Zb} + \mathbf{e}, \mathbf{z}_0'\mathbf{b} + e_0) = \mathbf{ZDz}_0 + \mathbf{w}$. Then $\mathbf{c}$ is substituted into Eq. (A.4.8):

$$\hat{y}_0 = x_0'\hat{a} + (ZDz_0 + w)'R^{-1}(y - X\hat{a} - Z\hat{b})$$

$$= x_0'\hat{a} + z_0'D(Z'R^{-1}y - Z'R^{-1}X\hat{a} - Z'R^{-1}Z\hat{b}) +$$

$$w'R^{-1}(y - X\hat{a} - Z\hat{b}). \quad (A.4.11)$$

From (A.2.2) we get:

$$Z'R^{-1}y - Z'R^{-1}X\hat{a} - Z'R^{-1}Z\hat{b} = D^{-1}\hat{b}. \quad (A.4.12)$$

Thus (A.4.11) simplifies

$$\hat{y}_0 = x_0'\hat{a} + z_0'\hat{b} + w'R^{-1}(y - X\hat{a} - Z\hat{b}). \quad (A.4.13)$$

Thus the prediction formula is exactly the same as if $\mathbf{b}$ were fixed; the only difference is in the way $\hat{\mathbf{b}}$ is computed.

Let us then derive $\text{var}(\hat{y}_0)$ and $\text{var}(\hat{y}_0 - y_0)$. Generally $\text{var}(\hat{y}_0)$ is not of interest but is necessary here when deriving an unbiased predictor in the arithmetic scale.

Define that

$$x_* = \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}, \quad (A.4.14)$$

$$X_* = [X\ Z], \text{ and} \quad (A.4.15)$$

$$a_* = \begin{bmatrix} a \\ b \end{bmatrix}. \quad (A.4.16)$$

Recall that $\text{var}(\hat{a}_* - a_*) = H^{-1}$, where H is given in (A.2.3). Consider the following partition of H^{-1}:

$$H^{-1} = \begin{bmatrix} \text{var}(\hat{a}) & \text{cov}(\hat{a},\hat{b}' - b') \\ \text{cov}(\hat{b} - b, a') & \text{var}(\hat{b} - b) \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \quad (A.4.17)$$

Let us first state some preliminary results. According to Henderson (1975):

$$\text{cov}(\hat{a},\hat{b}') = 0, \quad (A.4.18)$$

$$\text{var}(\hat{b}) = D - G_{22}, \text{ and} \quad (A.4.19)$$

$$\text{cov}(\hat{b}, b') = \text{var}(\hat{b}) = D - G_{22}. \quad (A.4.20)$$

From (A.4.19) and (A.4.20) it follows that

$$\text{cov}(\hat{b},\hat{b}' - b') = 0. \quad (A.4.21)$$

From (A.2.2) and (A.2.3) we see that

$\hat{a}_* = H^{-1}X_*'R^{-1}e$ + terms independent of e. Thus

$$\text{cov}(\hat{a}_*, e') = \text{cov}(\hat{a}_* - a_*, e') = H^{-1}X_*', \text{ and} \quad (A.4.22)$$

$$\text{cov}(\hat{a}_*, e_0) = \text{cov}(\hat{a}_* - a_*, e_0) = H^{-1}X_*'R^{-1}w. \quad (A.4.23)$$

Now:

$$\text{var}(\hat{y}_0) = \text{var}[x_0'\hat{a} + z_0'\hat{b} + w'R^{-1}(y - X\hat{a} - Z\hat{b})]$$

$$= \text{var}\{x_0'\hat{a} + z_0'\hat{b} + w'R^{-1}[-X(\hat{a} - a) - Z(\hat{b} - b) + e]\}.$$

Using Eqs. (A.4.17)—(A.4.22) and simple but tedious matrix algebra, this simplifies to:

$$\text{var}(\hat{y}_0) = x_0G_{11}x_0 + z_0'(D - G_{22})z_0 \quad (A.4.24)$$

$$- w'R^{-1}X_*H^{-1}X_*'R^{-1}w + w'R^{-1}w +$$

$$2z_0'G_{21}X'R^{-1}w + 2z_0'G_{22}Z'R^{-1}w.$$

The derivation of $\text{var}(\hat{y}_0 - y_0)$ is simpler. Write first:

$$\hat{y}_0 - y_0 = x_*'(\hat{a}_* - a_*) - e_0 + w'R^{-1}[-X_*(\hat{a}_* - a_*) + e]$$

$$= (x_*' - w'R^{-1}X_*)(\hat{a}_* - a_*) - e_0 + w'R^{-1}e.$$

From (A.4.22) and (A.4.23) it follows:

$$\text{var}(\hat{y}_0 - y_0) = (x_*' - w'R^{-1}X_*)H^{-1}(x_* - X_*'R^{-1}w) +$$

$$\text{var}(e_0) - w'R^{-1}w. \quad (A.4.25)$$

The first term of (A.4.25) represents the error variance due to the parameter estimation, and the difference $\text{var}(e_0) - w'R^{-1}w$ is the error variance when e_0 is regressed on e. When comparing (A.4.25) with the variance formula of Goldberger (1962), we see that the only effect of the randomness of b is that $X_*'R^{-1}X_*$ is replaced by H.

Computations can be simplified for special matrix structures, if the variance formulas are expressed in terms of submatrices of x_*, X_* and H^{-1}. Equation (A.4.25) can be written:

$$\text{var}(\hat{y}_0 - y_0) = x_0'G_{11}x_0 + z_0'G_{22}z_0 +$$

$$2x_0'G_{12}z_0 + w'R^{-1}X_*H^{-1}X_*'R^{-1}w$$

$$- 2x_0'G_{11}X'R^{-1}w - 2z_0'G_{21}X'R^{-1}w \quad (A.4.26)$$

$$- 2x_0'G_{12}Z'R^{-1}w - 2z_0'G_{22}Z'R^{-1}w +$$

$$\text{var}(e_0) - w'R^{-1}w$$

In (A.4.24) and (A.4.26) we can write further:

$$\mathbf{w}'\mathbf{R}^{-1}\mathbf{X}_*\mathbf{H}^{-1}\mathbf{X}_*'\mathbf{R}^{-1}\mathbf{w} = \mathbf{w}'\mathbf{R}^{-1}\mathbf{X}\mathbf{G}_{11}\mathbf{X}'\mathbf{R}^{-1}\mathbf{w} +$$

$$\mathbf{w}'\mathbf{R}^{-1}\mathbf{Z}\mathbf{G}_{22}\mathbf{Z}'\mathbf{R}^{-1}\mathbf{w} + 2\mathbf{w}'\mathbf{R}^{-1}\mathbf{X}\mathbf{G}_{12}\mathbf{Z}'\mathbf{R}^{-1}\mathbf{w}. \quad \text{(A.4.27)}$$

The predictors are also linear combinations of the observed y's. Thus the variance of the prediction error can also be partitioned into components in the same way as the variances of the parameter estimates. We do not utilize this decomposition in this study, however, but use instead an empirical descriptive model for the same purpose.

B. COMPUTATION OF MODEL PARAMETERS

B.1. Covariance components

Let us first consider the estimation of variance components, i.e., var[v(u)] and var[e(u)] for different angles u using the overall model. From the two equivalent formulations of the model we use the first one, namely (41.3). The estimation is achieved by means of the fitting constants method; consequently one needs to evaluate the terms of Eq. (A.3.5).

Let K denote the number of stands, n_k the number of trees in stand k, and N the total number of trees. Now the terms in (A.3.5) are as follows:

$$\mathbf{X} = \begin{bmatrix} 1 & s_{1,1} & s_{1,1}^2 & \bar{s}_{1.} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 1 & s_{K,n_K} & s_{K,n_K}^2 & \bar{s}_{K.} \end{bmatrix}, \quad \text{N}\times 4 \quad \text{(B.1.1)}$$

$$\text{rank}(\mathbf{X}) = 4,$$

$$\mathbf{a}' = [a_0(u), a_1(u), a_2(u), a_3(u)], \quad \text{(B.1.2)}$$

$$\mathbf{Z} = \begin{bmatrix} 1 & & \\ \vdots & & \\ 1 & & \\ & \ddots & \\ & & 1 \\ & & \vdots \\ & & 1 \end{bmatrix}, \quad \text{N}\times\text{K} \quad \text{(B.1.3)}$$

$$\text{rank}(\mathbf{Z}) = \text{K, and}$$

$$\mathbf{b}' = [v_1(u), \ldots, v_K(u)]. \quad \text{(B.1.4)}$$

The first and last column of $\mathbf{X}$ are linear combinations of the columns of $\mathbf{Z}$. So $\text{rank}(\mathbf{X}_*) = \text{rank}(\mathbf{X}) + \text{rank}(\mathbf{Z}) - 2 = \text{K}+2$ and $\text{rank}(\mathbf{X}_*) - \text{rank}(\mathbf{X}) = \text{K}-2$. Matrix $\mathbf{bb}'$ is now

$$\mathbf{bb}' = \begin{bmatrix} v_1(u)v_1(u) & \ldots & v_1(u)v_K(u) \\ \vdots & & \vdots \\ v_K(u)v_1(u) & \ldots & v_K(u)v_K(u) \end{bmatrix} \quad \text{(B.1.5)}$$

As $E[v_{k1}(u)v_{k2}(u)] = \text{var}[v(u)]$ if k1 = k2 and zero otherwise, $E(\mathbf{bb}') = \text{var}[v(u)]\mathbf{I}$. Thus

$$\text{tr}\{\mathbf{Z}'[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{Z}\, E(\mathbf{bb}')\} = \text{tr}\{\mathbf{Z}'[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{Z}\}\,\text{var}[v(u)] \quad \text{(B.1.6)}$$

Denote by C the coefficient of var[v(u)]. From the standard regression theory we know that $\mathbf{y}'[\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{y}$ is the sum of the squared residuals when y is regressed on $\mathbf{X}$. Thus the sum of the squared residuals is computed when each column of $\mathbf{Z}$ is regressed on $\mathbf{X}$. Then C is the total sum of these squared residuals.

The within-stand variance var[e(u)] is estimated by

$$\hat{\text{var}}[e(u)] = \text{SSE}(\mathbf{a},\mathbf{b})/[\text{N-rank}(\mathbf{X}_*)], \quad \text{(B.1.7)}$$

where SSE(**a**,**b**) is the sum of the squared residuals for the full model. Because the full model has a separate intercept for each stand, SSE(**a**,**b**) can be computed easily by regressing $d_{ki}(u)-\bar{d}_{k.}(u)$ on $s_{ki}-\bar{s}_{k.}$ and $s_{ki}^2-\bar{s}_{k.}^2$. Recall that SSE(**a**) is computed by regressing d(u) on $\mathbf{X}$. Thus var[v(u)] is estimated by:

$$\hat{var}[v(u)] = \{SSE(\mathbf{a})-SSE(\mathbf{a},\mathbf{b}) - \hat{var}[e(u)]\ [rank(\mathbf{X}_*)-rank(\mathbf{X})]\}/C. \quad (B.1.8)$$

As described in Appendix A.3, $cov[v(u_1), v(u_2)]$ and $cov[e(u_1), e(u_2)]$ can be estimated through the variance components of $d(u_1)$, $d(u_2)$ and $d(u_1) + d(u_2)$.

When the variance components are estimated for the regionalized model, the matrix $\mathbf{X}$ is block diagonal where the blocks are similar to $\mathbf{X}$ in the overall model. Hence $SSE(\mathbf{a},\mathbf{b})$, $SSE(\mathbf{a})$ and C can be computed for each region in the way described above. Then these regional quantities are added together. Let L be the number of regions. Then the matrix ranks for the regionalized model are:

$rank(\mathbf{X}) = 4L,$
$rank(\mathbf{Z}) = K,$ and
$rank(\mathbf{X}_*) = K + 2L.$

B.2. Parameters

Parameters of the model can be estimated when estimates of variance-covariance components are available. In fact, only the fixed parameters are of interest, although estimates of the random parameters will also be obtained. Note that the parameters can be estimated separately for each angle since the model (41.3) (or 41.4) has a separate set of parameters for each specific angle. Because the random stand and tree effects of different angles are correlated, the parameters can be estimated more efficiently by estimating them simultaneously for all angles (using 'seemingly unrelated regressions'). If the stand effects were treated as fixed parameters, then the simultaneous estimation would produce the same estimates as the separate estimation (see Johnston 1972, pp. 238—240).

In both separate and simultaneous estimation of parameters we are solving the system (A.2.2). For computational reasons, the parameters of the overall model and also the parameters of the regionalized model for regions with large number of stands in the data have been estimated separately for each angle. Note that the fixed parameters of the regionalized model can in any case be estimated separately for each region. In the separate estimation matrices $\mathbf{X}$ and $\mathbf{Z}$ and vectors $\mathbf{a}$ and $\mathbf{b}$ in equation (A.2.2) are as defined in B.1. Furthermore,

$$\mathbf{R} = W(u)\mathbf{I}_N, \text{ and} \quad (B.2.1)$$
$$\mathbf{D} = B(u)\mathbf{I}_K, \quad (B.2.2)$$

where $W(u)=var[e(u)]$, and $B(u)=var[v(u)]$.

When the parameters are estimated separately for each knot angle, only estimates of the variance components are used. Thus the definiteness of the estimated covariance matrices causes no problem, and we can use directly the variance component estimates obtained by means of the fitting constants method when solving the system (A.2.2).

For the regionalized model, the simultaneous estimation is computationally feasible for regions with small number of stands and trees. The quantities in the equation (A.2.2) are as follows:

$$\mathbf{X} = \begin{bmatrix} \mathbf{I} & s_{1,1}\mathbf{I} & s^2_{1,1}\mathbf{I} & \bar{s}_{1.}\mathbf{I} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{I} & s_{K,n_K}\mathbf{I} & s^2_{K,n_K}\mathbf{I} & \bar{s}_{K.}\mathbf{I} \end{bmatrix}_{14N\times 56} \quad (B.2.3)$$

where $\mathbf{I}$ is the identity matrix of order 14,

$$\mathbf{a}' = [a_0(1),...,a_0(14),a_1(1),...,a_2(1),...,a_3(1),...,a_3(14)] \quad (B.2.4)$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{I} & & \\ \vdots & & \\ \mathbf{I} & \cdots & \\ & & \mathbf{I} \\ & & \vdots \\ & & \mathbf{I} \end{bmatrix}_{14N\times 14K} \quad (B.2.5)$$

$$\mathbf{b}' = [v_1(1),..., v_1(14), v_2(1),..., v_K(14)], \quad (B.2.6)$$

$$\mathbf{R} = diag(\mathbf{W}, ..., \mathbf{W})_{14N\times 14N}, \quad (B.2.7)$$

$$\mathbf{D} = diag(\mathbf{B}, ..., \mathbf{B})_{14K\times 14K}. \quad (B.2.8)$$

As described in section 43, the matrix $\mathbf{B}$ in (B.2.8) is obtained from the fitting constants estimate by adding a small constant to the diagonal. Some attention should, however, also be paid to the rank of $\mathbf{R}$. It was claimed in section 44 that a 13×13 submatrix of $\mathbf{W}$ is, in theory, singular; thus $\mathbf{W}$ and $\mathbf{R}$ are also in theory singular. Consequently, the dimensionality of $\mathbf{W}$ should be decreased, e.g., by expressing one diameter as a

linear combination of the other diameters before applying Eq. (A.2.2), where **R** (and thus also **W**) is to be inverted. Because the fitting constants estimate of **W** was slightly (but clearly) positive definite, the estimation was accomplished using the actual estimate of **W**.

C. COMPUTATIONS IN THE APPLICATIONS

C.1. Parameter estimates

In applications, according to (52.5) the model is:

$$y(u_{ij}) = a(u_{ij})s_i + \sum_{k=1}^{p} q_k(u_{ij})c_k + e(u_{ij}).$$

The model is in a form where the standard procedure of estimation for mixed linear models can be applied. Let n and m_i be the number of trees in the stand and the number of measurements for tree i, respectively. The terms of the general model $\mathbf{y}=\mathbf{Xa}+\mathbf{Zb}+\mathbf{e}$ are interpreted as follows:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \text{ where} \tag{C.1.1}$$

$$\mathbf{y}_i' = [y(u_{i1}),...,y(u_{i,m_i})], \tag{C.1.2}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{a}_1 & & \\ & \ddots & \\ & & \mathbf{a}_n \end{bmatrix}, \text{ where} \tag{C.1.3}$$

$$\mathbf{a}_i' = [a(u_{i1}),..., a(u_{i,m_i})], \tag{C.1.4}$$

$$\mathbf{a}' = \mathbf{s}' = (s_1,..., s_n), \tag{C.1.5}$$

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_1 \\ \vdots \\ \mathbf{Z}_n \end{bmatrix}, \text{ where} \tag{C.1.6}$$

$$\mathbf{Z}_i = \begin{bmatrix} q_1(u_{i1}) & \dots & q_p(u_{i1}) \\ \vdots & & \vdots \\ q_1(u_{i,m_i}) & \dots & q_p(u_{i,m_i}) \end{bmatrix}, \tag{C.1.7}$$

$$\mathbf{b}' = \mathbf{c}' = (c_1, ..., c_p), \text{ and} \tag{C.1.8}$$

$$\mathbf{e}' = [e(u_{11}), ..., e(u_{n,m_n})] \tag{C.1.9}$$

In the estimation the covariance matrices $\mathbf{D} = \text{var}(\mathbf{b})$ and $\mathbf{R}=\text{var}(\mathbf{e})$ are also needed. In this case:

$$\mathbf{D} = \text{diag}(t_1, ..., t_p), \tag{C.1.10}$$

where t_k, k=1,...,p, is the k^{th} characteristic root of **B**, the covariance matrix of the stand effects v(u). The matrix **R** is now of the form:

$$\mathbf{R} = \text{diag}(\mathbf{R}_1, ..., \mathbf{R}_n), \text{ where} \tag{C.1.11}$$

$$\mathbf{R}_i = \text{var}[e(u_{i1}), ..., e(u_{i,m_i})] \tag{C.1.12}$$

Matrix **R** is thus block-diagonal having as many blocks as there are measured trees, and the dimension of each block is equal to the number of measurements made for the respective tree. The elements of **R** are computed from the within-stand covariance matrix **W** using two-dimensional splines.

When the principal components of the between stand effects were computed, the crown height was excluded. Hence v(14) cannot be expressed in terms of c_k, k=1,...,p, without error. If the crown height is measured for some trees, then we must add a new random effect $c_{p+1}=v(14)$ to the vector of random effects. In addition, the **Z**-matrix must be modified accordingly. The row of the **Z**-matrix corresponding to the measurement j on tree i, $\mathbf{z}(u_{ij})'$, becomes:

$$\mathbf{z}(u_{ij})' = [q_1(u_{ij}), ..., q_p(u_{ij}), 0] \tag{C.1.13}$$

if the measurement is not the crown height, and

$$\mathbf{z}(u_{ij})' = (0, ..., 0, 1) \tag{C.1.14}$$

for the crown height. **D**-matrix becomes

$$\mathbf{D} = \begin{bmatrix} t_1 & & & \text{cov}[c_1,v(14)] \\ & \ddots & & \vdots \\ & & t_p & \text{cov}[c_p,v(14)] \\ \text{cov}[v(14),c_1] & \dots & \text{cov}[v(14),c_p] & \text{var}[v(14)] \end{bmatrix}. \tag{C.1.15}$$

Covariance $\text{cov}(c_k,v(14))$ is obtained by recalling that (see 52.1)

$$c_k = \sum_{u=1}^{13} q_k(u)v(u), \text{ thus}$$

$$\text{cov}[c_k,v(14)] = \sum_{u=1}^{13} q_k(u)B(u,14). \qquad \text{(C.1.16)}$$

Computations can be simplified considerably using the sparsity of **X**- and **R**-matrices. First $\mathbf{X'R^{-1}}$ is computed:

$$\mathbf{X'R^{-1}} = \begin{bmatrix} \mathbf{a}_1'\mathbf{R}_1^{-1} & & \\ & \ddots & \\ & & \mathbf{a}_n'\mathbf{R}_n^{-1} \end{bmatrix}. \qquad \text{(C.1.17)}$$

Matrix $\mathbf{X'R^{-1}X}$ is diagonal, the i^{th} diagonal element being

$$(\mathbf{X'R^{-1}X})_{ii} = \mathbf{a}_i'\mathbf{R}_i^{-1}\mathbf{a}_i. \qquad \text{(C.1.18)}$$

$\mathbf{X'R^{-1}Z}$ and $\mathbf{X'R^{-1}y}$ can also be computed treewise:

$$\mathbf{X'R^{-1}Z} = \begin{bmatrix} \mathbf{a}_1'\mathbf{R}_1^{-1}\mathbf{Z}_1 \\ \vdots \\ \mathbf{a}_n'\mathbf{R}_n^{-1}\mathbf{Z}_n \end{bmatrix}, \text{ and} \qquad \text{(C.1.19)}$$

$$\mathbf{X'R^{-1}y} = \begin{bmatrix} \mathbf{a}_1'\mathbf{R}_1^{-1}\mathbf{y}_1 \\ \vdots \\ \mathbf{a}_n'\mathbf{R}_n^{-1}\mathbf{y}_n \end{bmatrix}. \qquad \text{(C.1.20)}$$

The block diagonal form of **R** simplifies naturally the computation of $\mathbf{Z'R^{-1}Z}$ and $\mathbf{Z'R^{-1}y}$. Because $\mathbf{X'R^{-1}X}$ (left upper corner of **H** in (A.2.3) is diagonal, matrix **H** can easily be inverted using Cholesky decomposition (see, e.g., Kennedy and Gentle 1980) also when the number of trees is large.

Let us now consider briefly the estimation when sizes of trees, s_i's, are assumed to be independent random parameters with mean $\bar{s}_{..}$ and variance var(s). The estimation procedure for random parameters assumes that the expectations are zero. Equation (52.5) can be rewritten as follows:

$$y(u_{ij}) - a(u_{ij})\bar{s}_{..} = \qquad \text{(C.1.21)}$$

$$a(u_{ij})(s_i - \bar{s}_{..}) + \sum_{k=1}^{p} q_k(u_{ij})c_k + e(u_{ij}).$$

Then $y(u_{ij}) - a(u_{ij})\bar{s}_{..}$ is the dependent variable and $(s_i - \bar{s}_{..})$ is the random size parameter to be estimated. For fixed size, $\mathbf{X'R^{-1}X}$ was found to be diagonal with diagonal elements $\mathbf{a}_i'\mathbf{R}_i^{-1}\mathbf{a}_i$ (C.1.18). If the s_i's are now treated as independent random parameters, $\text{var}(s)^{-1}$ must be added to $\mathbf{a}_i'\mathbf{R}_i^{-1}\mathbf{a}_i$. Otherwise the estimation proceeds as for fixed s.

The standard estimation theory can also be applied when s is assumed to have a random or fixed stand effect and a random within-stand effect. In this case we should re-evaluate the treatment of $\bar{s}_{k.}$ in the estimation process.

C.2. Predicting the logarithmic stem curve

We have now estimated the size of tree i, s_i, and the first p principal components (c_k, k=1,...,p) of the random stand effects. The problem is then to predict the stem curve of tree i at knot angles u=1,...,13. The main task is to predict the y-variable defined in (51.4) at knot angles. The predictor of y for tree i at the knot angles is computed using prediction formula (A.4.13) derived in the Appendix A.4:

$$\hat{y}_0 = \mathbf{x}_0'\hat{\mathbf{a}} + \mathbf{z}_0'\hat{\mathbf{b}} + \mathbf{w}'\mathbf{R}^{-1}(\mathbf{y} - \mathbf{X}\hat{\mathbf{a}} - \mathbf{Z}\hat{\mathbf{b}}).$$

y, X, a, Z, b and **R** are given for this special case in Appendix C.1. Other variables have the following interpretations:

$$\hat{y}_0 = \hat{y}_i(u), \qquad \text{(C.2.1)}$$

$$\mathbf{x}_0' = [0,..., 0, a(u), 0, ..., 0], \qquad \text{(C.2.2)}$$

$$\mathbf{z}_0' = \mathbf{z}(u)' = [q_1(u), ..., q_p(u)], \qquad \text{(C.2.3)}$$

$$\mathbf{w}' = [0, ..., 0, \mathbf{w}_i'(u), 0, ..., 0], \text{ where} \qquad \text{(C.2.4)}$$

$$\mathbf{w}_i' = [W(u,u_{i1}), ..., W(u,u_{i,m_i})]. \qquad \text{(C.2.5)}$$

Computations can be simplified by noting that:

$$\mathbf{x}_0'\hat{\mathbf{a}} = a(u)\hat{s}_i \qquad \text{(C.2.6)}$$

and

$$\mathbf{w}'\mathbf{R}^{-1} = [0, ..., 0, \mathbf{w}_i'\mathbf{R}_i^{-1}, 0,..., 0]. \qquad \text{(C.2.7)}$$

The prediction formula can be written as:

$$\hat{y}_i(u) = a(u)\hat{s}_i + \mathbf{z}(u)'\hat{\mathbf{c}} + \mathbf{w}_i'\mathbf{R}_i^{-1}(\mathbf{y}_i - \hat{s}_i\mathbf{a}_i - \mathbf{Z}_i\hat{\mathbf{c}}). \qquad \text{(C.2.8)}$$

Because u in $W(u,u_{ij}) = \text{cov}[e(u), e(u_{ij})]$ is a knot angle, $W(u,u_{ij})$ can be computed from the within-stand covariance matrix **W** using one-dimensional splines. If the crown height has been measured for some trees, a random stand effect for the crown height

(c_{p+1}) is used in the last term but can be omitted from $z(u)'\hat{c}$, because its coefficient is zero (see Appendix C.1). The predicted log-diameter $\hat{d}_i(u)$ is obtained from $\hat{y}_i(u)$ using (53.1).

When $\mathrm{var}[\hat{y}_i(u)]$ and $\mathrm{var}[\hat{y}_i(u)-y_i(u)]$ are computed by means of Eqs. (A.4.24), (A.4.26) and (A.4.27), all terms containing $\mathbf{w}$ or $\mathbf{x}_0$ can be simplified by making the following substitutions:

$$\mathbf{x}_0 \rightarrow a(u) \tag{C.2.9}$$
$$\mathbf{G}_{11} = \mathrm{var}(\hat{\mathbf{a}}) \rightarrow \mathrm{var}(\hat{s}_i) \tag{C.2.10}$$
$$\mathbf{w}'\mathbf{R}^{-1}\mathbf{w} \rightarrow \mathbf{w}_i'\mathbf{R}_i^{-1}\mathbf{w}_i \tag{C.2.11}$$
$$\mathbf{G}_{21} = \mathrm{cov}(\hat{\mathbf{b}}-\mathbf{b},\hat{\mathbf{a}}') \rightarrow \mathrm{cov}(\hat{c}-\mathbf{c},\hat{s}_i) \tag{C.2.12}$$
$$\mathbf{X}'\mathbf{R}^{-1}\mathbf{w} \rightarrow \mathbf{a}_i'\mathbf{R}_i^{-1}\mathbf{w}_i \tag{C.2.13}$$
$$\mathbf{Z}'\mathbf{R}^{-1}\mathbf{w} \rightarrow \mathbf{Z}_i'\mathbf{R}_i^{-1}\mathbf{w}_i \tag{C.2.14}$$

Note that (C.2.11) and (C.2.14) are, in fact, true identities. Note also that $\mathbf{z}_0'\mathbf{D}\mathbf{z}_0$ and $\mathbf{z}_0'\mathbf{G}_{22}\mathbf{z}_0$ in (A.4.24) and (A.4.26) need to be computed only once for a given stand.

At the application stage all the needed variances and covariances are assumed to be known. This means that the prediction variance is also known.

C.3. Formulas for trees with one measured dimension

Suppose the model has been formulated for n trees as described above. Then, add to the data a tree with a single measurement. Denote this as observation number zero, the corresponding measurement angle being u_{01}. Then the quantities in the corresponding model $\mathbf{y}=\mathbf{X}\mathbf{a}+\mathbf{Z}\mathbf{b}+\mathbf{e}$ are as follows:

$$\mathbf{y} = \begin{bmatrix} y(u_{01}) \\ \mathbf{y}_* \end{bmatrix}, \tag{C.3.1}$$

$$\mathbf{X} = \begin{bmatrix} a(u_{01}) & \\ & \mathbf{X}_* \end{bmatrix}, \tag{C.3.2}$$

$$\mathbf{a} = \begin{bmatrix} s_0 \\ \mathbf{s}_* \end{bmatrix}, \tag{C.3.3}$$

$$\mathbf{Z} = \begin{bmatrix} z(u_{01})' \\ \mathbf{Z}_* \end{bmatrix}, \tag{C.3.4}$$

$$\mathbf{e} = \begin{bmatrix} e(u_{01}) \\ \mathbf{e}_* \end{bmatrix}, \tag{C.3.5}$$

$$\mathbf{R} = \mathrm{var}(\mathbf{e}) = \begin{bmatrix} W(u_{01}) & \\ & \mathbf{R}_* \end{bmatrix}, \tag{C.3.6}$$

where $\mathbf{y}_*$, $\mathbf{X}_*$, $\mathbf{s}_*$, $\mathbf{e}_*$ and $\mathbf{R}_*$ are the corresponding quantities without the new observation; $\mathbf{b}=\mathbf{c}$ and $\mathbf{D}=\mathrm{var}(\mathbf{c})$ are as before. Then $\hat{\mathbf{c}}$ can be computed using (A.2.6). For that, $\mathbf{S}$ defined in (A.2.7) needs to be computed first. In this case direct matrix multiplication gives:

$$\mathbf{S} = \begin{bmatrix} 0 & \\ & \mathbf{S}_* \end{bmatrix}, \tag{C.3.7}$$

where $\mathbf{S}_*$ is the $\mathbf{S}$ matrix of (A.2.7) without the new observation. When (A.2.6) is used to compute $\hat{\mathbf{c}}$, we find that $\hat{\mathbf{c}}$ is the same with and without the new observation. Thereafter, the size vector $\mathbf{s}$ can be computed using (A.2.8). Note that the new observation does not affect the estimation of the sizes of the previous trees ($\mathbf{s}_*$) and s_0 is estimated by

$$\hat{s}_0 = a(u_{01})^{-1}[y(u_{01})-z(u_{01})'\hat{c}]. \tag{C.3.8}$$

That is, s_0 is solved from the estimated stand stem curve:

$$y(u_{01}) = a(u_{01})s_0 + z(u_{01})'\hat{c}.$$

When other dimensions $y_0(u)$ of the new tree are predicted, in the general prediction formula (A.4.13)

$$\hat{y} = \mathbf{x}_0'\hat{\mathbf{a}} + \mathbf{z}_0'\hat{\mathbf{b}} + \mathbf{w}'\mathbf{R}^{-1}(\mathbf{y}-\mathbf{X}\hat{\mathbf{a}}-\mathbf{Z}\hat{\mathbf{b}})$$

the only nonzero component of $\mathbf{w}$ is the first one; on the other hand, the first component of $\mathbf{y}-\mathbf{X}\hat{\mathbf{a}}-\mathbf{Z}\hat{\mathbf{b}}$ is zero. Thus the last term can be ignored, and the stand stem curve should be used to predict the other dimension $y_0(u)$:

$$\hat{y}_0(u) = a(u)\hat{s}_0 + z(u)'\hat{c}. \tag{C.3.9}$$

It is easier to derive $\mathrm{var}[\hat{y}_0(u)]$ and $\mathrm{var}[\hat{y}_0(u)-y_0(u)]$ directly instead of using the general matrix formulas of Appendix

A.4. Let us derive first some preliminary results; according to (C.3.8) $\hat{s}_0$ can be written:

$$\hat{s}_0 = a(u_{01})^{-1}[y(u_{01}) - \mathbf{z}(u_{01})'\hat{\mathbf{c}}]$$

$$= a(u_{01})^{-1}[a(u_{01})s_0 - \mathbf{z}(u_{01})'(\hat{\mathbf{c}} - \mathbf{c}) + e(u_{01})]. \quad \text{(C.3.10)}$$

Note that

$$\text{cov}[\hat{\mathbf{c}} - \mathbf{c}, e(u_{01})] = 0. \quad \text{(C.3.11)}$$

According to (A.4.17), denote:

$$\mathbf{G}_{22} = \text{var}(\hat{\mathbf{c}} - \mathbf{c}).$$

Then we get from (C.3.10) and (C.3.11):

$$\text{var}(\hat{s}_0) = a(u_{01})^{-2}[\mathbf{z}(u_{01})'\mathbf{G}_{22}\mathbf{z}(u_{01}) + W(u_{01})], \quad \text{(C.3.12)}$$

$$\text{cov}(\hat{s}_0, \hat{\mathbf{c}}' - \mathbf{c}') = -a(u_{01})^{-1}\mathbf{z}(u_{01})'\mathbf{G}_{22} \quad \text{(C.3.13)}$$

$$\text{cov}[\hat{s}_0, e_0(u)] = a(u_{01})^{-1}W(u_{01}, u). \quad \text{(C.3.14)}$$

According to (A.4.18) and (A.4.19)

$$\text{cov}(\hat{s}_0, \hat{\mathbf{c}}) = 0, \text{ and} \quad \text{(C.3.15)}$$

$$\text{var}(\hat{\mathbf{c}}) = \mathbf{D} - \text{var}(\hat{\mathbf{c}} - \mathbf{c}) = \mathbf{D} - \mathbf{G}_{22}. \quad \text{(C.3.16)}$$

Let

$$\alpha = a(u)/a(u_{01}). \quad \text{(C.3.17)}$$

From (C.3.12), (C.3.15) and (C.3.16) we get

$$\text{var}[\hat{y}_0(u)] = \text{var}[a(u)\hat{s}_0 + \mathbf{z}(u)'\hat{\mathbf{c}}]$$
$$= a(u)^2\text{var}(\hat{s}_0) + \mathbf{z}(u)'\text{var}(\hat{\mathbf{c}})\mathbf{z}(u)$$
$$= \alpha^2[\mathbf{z}(u_{01})'\mathbf{G}_{22}\mathbf{z}(u_{01}) + W(u_{01})] +$$
$$\mathbf{z}(u)'(\mathbf{D} - \mathbf{G}_{22})\mathbf{z}(u). \quad \text{(C.3.18)}$$

The variance of the prediction error is obtained from (C.3.11)—(C.3.14):

$$\text{var}[\hat{y}_0(u) - y_0(u)] =$$

$$\text{var}[a(u)\hat{s}_0 + \mathbf{z}(u)'\hat{\mathbf{c}} - a(u)s_0 - \mathbf{z}(u)'\mathbf{c} - e_0(u)]$$
$$= \text{var}[a(u)\hat{s}_0 + \mathbf{z}(u)'(\hat{\mathbf{c}} - \mathbf{c}) - e_0(u)]$$
$$= W(u) + \alpha^2 W(u_{01}) - 2\alpha W(u_{01}, u) +$$
$$[\alpha\mathbf{z}(u_{01})' - \mathbf{z}(u)']\mathbf{G}_{22}[\alpha\mathbf{z}(u_{01}) - \mathbf{z}(u)]. \quad \text{(C.3.19)}$$

If we have measured only one dimension for each tree in the stand, we must use the population stem curve to estimate the stem curves of the trees. The above formulas can also be applied in this case by taking $\hat{\mathbf{c}} = 0$ and $\mathbf{G}_{22} = \text{var}(\hat{\mathbf{c}} - \mathbf{c}) = \text{var}(\mathbf{c})$.

If tree sizes were assumed to be random parameters, then trees with one measured dimension contain information on the stem form and cannot be estimated separately. For instance, if we know that trees cannot be very big, then we know that a tree with a very large diameter at breast height is probably relatively thick. The distinction between fixed and random size may become clearer if we derive the estimators for the size and stand effects when only one tree with one measured dimension is measured in a stand. Suppose that tree sizes are random with mean $\bar{s}_{..}$ and variance var(s). Then

$$y(u_{01}) = a(u_{01})s_0 + \mathbf{z}(u_{01})'\mathbf{c} + e(u_{01}), \quad \text{(C.3.20)}$$

$$\text{cov}[s_0, y(u_{01})] = a(u_{01})\text{var}(s), \text{ and} \quad \text{(C.3.21)}$$

$$\text{var}[y(u_{01})] = a(u_{01})^2\text{var}(s) + \mathbf{z}(u_{01})'\mathbf{D}\mathbf{z}(u_{01}) + W(u_{01}). \quad \text{(C.3.22)}$$

The standard regression estimator (predictor) for s_0 is then obtained from:

$$\hat{s}_0 - \bar{s}_{..} =$$

$$\frac{y(u_{01})/a(u_{01}) - \bar{s}_{..}}{1 + [\mathbf{z}(u_{01})'\mathbf{D}\mathbf{z}(u_{01}) + W(u_{01})]/[a(u_{01})\text{var}(s)]} \quad \text{(C.3.23)}$$

If the size is fixed, then s_0 is estimated by $y(u_{01})/a(u_{01})$. Thus the effect of the randomness of size is that the estimate is shifted towards the mean size, the amount of shift being dependent on

$$[\mathbf{z}(u_{01})'\mathbf{D}\mathbf{z}(u_{01}) + W(u_{01})]/[a(u_{01})\text{var}(s)].$$

As var(s) tends to infinity, the estimator of fixed size is the limiting estimator of random size. The standard regression predictor for the random stand effects is:

$$\hat{\mathbf{c}} = \text{cov}[\mathbf{c}, y(u_{01})][y(u_{01}) - a(u_{01})\bar{s}_{..}]/\text{var}[y(u_{01})], \quad \text{(C.3.24)}$$

where $\text{var}[y(u_{01})]$ is given in (C.3.22), and $\text{cov}[\mathbf{c}, y(u_{01})] = \mathbf{D}\mathbf{z}(u_{01})$.

Thus in the case of random size, already with one measurement in the stand, we generally get nonzero estimates for the stand effects.

D. LIST OF SYMBOLS

$\mathbf{a}$ = a vector of fixed parameters

$a_0(u), a_1(u), a_2(u), a_3(u)$ = parameters of the stem form model, in the applications they are treated as variables.

$a(u)$ = parameter (variable) for angle u, used in the applications and obtained using $a_1(u)$, $a_2(u)$ and a preliminary estimate of the size of the tree ($\dot{s}$).

$\mathbf{B}$ = covariance matrix of the random stand effects at the knot angles.

$B(u_1,u_2)$ = covariance between the stand effects at angles u_1 and u_2, obtained from $\mathbf{B}$ by interpolation.

$\mathbf{b}$ = a vector of random parameters.

$\mathbf{c}=(c_1,\ldots,c_p)$ = vector of the first p principal components of the random stand effects; in applications this is the vector of random parameters to be estimated.

$\mathbf{D}$ = covariance matrix of the random parameters (between-class effects) in a linear model.

$D(u)$ = diameter (cm) at the angle u in the polar coordinate system.

$d(u)$ = $\ln[D(u)]$.

$D_{1.3}$ = diameter over bark at the height of 1.3 m above ground level, i.e., diameter at breast height.

D_6 = diameter at the height of 6 m.

$e_{ki}(u)$ = random tree effect of the log-diameter at angle u for tree i in stand k.

$e_m(u)$ = random measurement error at angle u.

$\mathbf{H}$ = coefficient matrix in a mixed linear system.

H = tree height (m).

H_c = crown height (m).

$\mathbf{I}$ = identity matrix.

i = index for trees.

j = index for measurements.

k = index for stands.

m = number of measurements for a given tree.

N = number of observations (measurements).

n = number of trees in a stand.

p = the (assumed) rank or the between-stand covariance matrix or the number of principal components of the random stand effects used in the estimation.

$\mathbf{q}_k$ = k^{th} characteristic vector of the between-stand covariance matrix $\mathbf{B}$.

$\mathbf{Q}$ = $(\mathbf{q}_1 \ldots \mathbf{q}_{13})'$.

$\mathbf{R}$ = N×N covariance matrix of within-class effects.

$R(u)$ = ray to the stem curve at angle u in the polar coordinate system.

$r(u)$ = $\ln[R(u)]$.

S = size of a tree in the arithmetic scale, e^s.

s = size of a tree defined as a weighted mean of logarithmic diameters; an artificial fixed variable in the analysis stage, and a fixed or random parameter in the applications.

$\dot{s}$ = preliminary estimate of s, used in the applications.

$\bar{s}$ = average size of trees belonging to same stand.

s_b, s_w = empirically estimated between-stand and within-stand standard errors of estimates.

σ_b, σ_w = theoretically derived between-stand and within-stand standard errors of estimates.

t_k = k^{th} characteristic root (eigenvalue) of the between-stand covariance matrix $\mathbf{B}$.

u = angle in the polar coordinate system, integer values u=1,...,13 are used for the knot angles; u=14 is for the crown height.

u_{ij} = angle in the j^{th} measurement for tree i.

$\mathbf{V}$ = covariance matrix of random effects (including both within-class and between-class effects) of a linear model.

V = volume of a tree stem (dm^3).

$v(u)$ = random stand effect at angle u.

$\mathbf{W}$ = covariance matrix of the tree effects at the knot angles.

$W(u_1,u_2)$ = covariance between the tree effects at angles u_1 and u_2, obtained from $\mathbf{W}$ by interpolation.

$\mathbf{w}$ = vector of the covariances between the random tree effects of observed dimensions and a dimension to be predicted.

$\mathbf{X}$ = model ('design') matrix for the fixed effects.

y = dependent variable, in the applications y(u) is closely related to the log-diameter at angle u.

$\mathbf{Z}$ = model ('design') matrix of the random effects in a linear model.

$\mathbf{z}$ = vector of regressors having random coefficients, in applications the k^{th} element of $\mathbf{z}(u)$ is obtained from the k^{th} characteristic vector $\mathbf{q}_k$ by interpolation.

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The model presented provides a compact general description of how the stem form of Scots pine varies as a multidimensional object. Stem dimensions are determined in polar coordinates. Logarithmic dimensions are analyzed by a linear model in which the size and relative size of a tree are fixed explanatory variables. The random variation in stem dimensions is partitioned into variation between stands and variation within stands. The between-stand and within-stand covariance matrices are analyzed by principal components. In applications, tree sizes are parameters to be estimated, and parameters of the stem form model are known variables. The stem curve and volume of a tree can be estimated if any stem dimensions are measured. The stem curve can be calibrated for a given stand by estimating the random stand effects. The model can be used to study different measurement strategies. Measurement errors can be taken into account in the estimation.

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The model presented provides a compact general description of how the stem form of Scots pine varies as a multidimensional object. Stem dimensions are determined in polar coordinates. Logarithmic dimensions are analyzed by a linear model in which the size and relative size of a tree are fixed explanatory variables. The random variation in stem dimensions is partitioned into variation between stands and variation within stands. The between-stand and within-stand covariance matrices are analyzed by principal components. In applications, tree sizes are parameters to be estimated, and parameters of the stem form model are known variables. The stem curve and volume of a tree can be estimated if any stem dimensions are measured. The stem curve can be calibrated for a given stand by estimating the random stand effects. The model can be used to study different measurement strategies. Measurement errors can be taken into account in the estimation.

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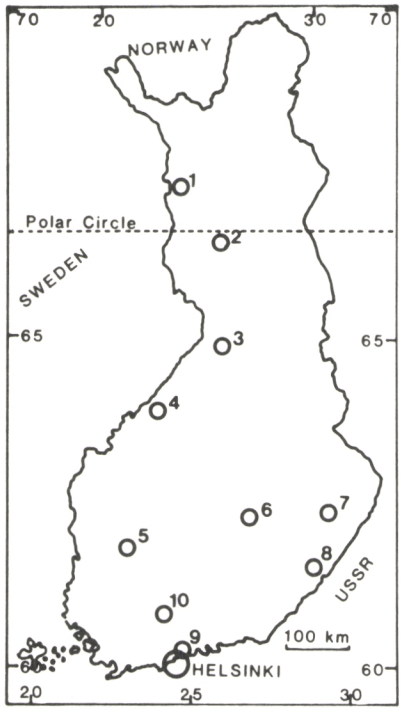

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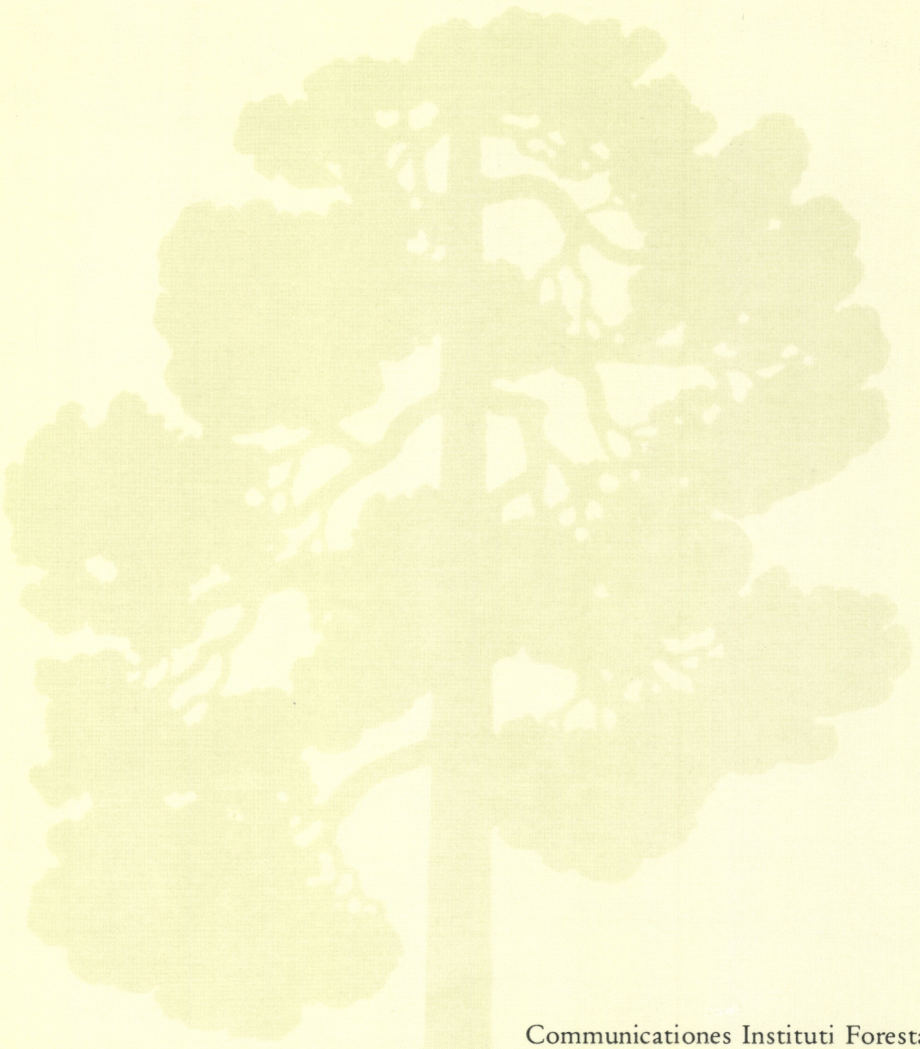
8 Punkaharju
9 Ruotsinkylä
10 Ojajoki

FACTS ABOUT FINLAND

Total land area: 304 642 km^2 of which 60—70 per cent is forest land.

Mean temperature, °C:	Helsinki	Joensuu	Rovaniemi
January	-6,8	-10,2	-11,0
July	17,1	17,1	15,3
annual	4,4	2,9	0,8
Thermal winter (mean temp. < 0°C):	20.11.—4.4.	5.11.—10.4.	18.10.—21.4.

Most common tree species: Pinus sylvestris, Picea abies, Betula pendula, Betula pubescens



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