

# **Optimal forest rotation with *in situ* preferences**

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This study develops a utility maximization, infinite horizon forest rotation model that includes *in situ* values and the forest owner's consumption/savings decision making. The value of forest land becomes owner-specific and depends on property rights related to *in situ* benefits. The length of the rotation period always depends on the initial stand age and the wealth of the forest owner. Even if the forest owner's subjective time preference equals the rate of interest, the rotation period may evolve in time and it may be optimal to give up forest harvesting in finite time. A forest owner with accumulating nonforest assets never continues harvesting forever. In contrast, a forest owner with decreasing nonforest assets adjusts the rotation period toward the Faustmann solution. In a multiple stand version of the model, the harvesting decisions regarding different stands are linked together via the budget constraint and the dependence of an individual stand's *in situ* value on the age structure of the whole forest. Numerical simulation with two stands suggests that within the class of concave *in situ* valuation functions the optimal solution yields convergence toward forests with increasing heterogeneity of age structures.

Metsään liitetyt *in situ* (ei-puuntuotannolliset) arvostukset tuottavat monia puun tarjontaa, metsämaan arvoa ja taloudellisia ohjauskeinoja koskevia avoimia kysymyksiä. Lisäksi voidaan yleisemmin kysyä miten *in situ* arvostukset muuttavat taloudellisesti tehokkaan puuntuotannon ja esimerkiksi kiertoajan määrätymisen peruseräitä. Näiden kysymysten tarkastelemiseksi tarvitaan talousteoreettisesti perusteltu ja empiirisesti validi metsänomistajien rationaaliseen käyttäytymiseen perustuva mallikehikko.

Taloudellinen yksityismetsänomistajien puuntarjontatutkimus sisältää kaksi erilaista tutkimustraditiota, joihin metsän *in situ* arvostusten tutkimus voidaan perustaa. Nämä ovat Pohjoismaissa kehitetty kahden periodin metsämalli ja Pohjois-Amerikassa sovellettu Faustmannin rotaatiomalliin perustuva tutkimustraditio. Kahden periodin mallin etuna on sen talousteoreettinen yleisyys. Mallin avulla voidaan esimerkiksi tarkastella miten epätäydelliset pääomamarkkinat, epävarmuus ja *in situ* arvostukset vaikuttavat puun tarjontaan. Ongelmana on kuitenkin malliin sisältyvä kuvaus metsästä homogeenisena biomassana ilman ikäluokkia. Kahden periodin mallit eivät tämän seurauksena voi tuottaa esimerkiksi rotaatioperiodin pituutta koskevia tuloksia. Faustmannin malliin perustuvan tradition etuna on todellisuutta paremmin vastaava kuvaus tasaikäisiin puustoihin perustuvasta metsänhoidosta. Ongelmana on kuitenkin metsänomistajakohtaisten tekijöiden jääminen mallin ulkopuolelle sellaisissakin tapauksissa, joissa tämä ei ole kahden periodin tradition tulosten nojalla mahdollista. Erityisesti tämä ongelma koskee tapausta, jossa tutkitaan *in situ* arvostusten vaikutusta puun tarjontaan.

Tässä tutkimuksessa yhdistetään edellä mainittujen traditioiden vahvuudet kehittämällä yksityismetsänomistajan päätöksentekoa kuvaava hyödyn maksimointiin perustuva optimirotaatiomalli, joka sisältää metsänomistajakohtaiset tekijät ja metsään liitetyt *in situ* arvostukset. Metsänomistajalla oletetaan olevan altruistinen perinnönjättömotiivi, pääomamarkkinoiden oletetaan olevan täydelliset ja *in situ* hyötyjen oletetaan riippuvan puuston iästä tai usean puuston tapauksessa puustoista koostuvan metsän ikäluokkarakenteesta.

Tutkimuksen päätulokset ovat seuraavat: ainoastaan erikoistapauksessa optimikiertoajan pituus on vakio ja hakkuukertojen määrä ääretön kuten Faustmannin mallissa. Varallisuuttaan kasvattavilla metsänomistajilla kiertoaika ajan kuluessa pitenee ja hakkuukertojen määrä on aina äärellinen. Varallisuuttaan kuluttavilla metsänomistajilla kiertoaika lyhenee ja lähestyy Faustmannin kiertoaikaa. Kiertoajan pituus riippuu aina metsänomistajan varallisuudesta, puuston iästä tarkasteluajan alussa ja metsänomistajan

subjektiivisista arvoista ja arvostuksista. *In situ* arvostukset pidentävät puuston kiertoaikaa, jos *in situ* arvo kasvaa puuston iän funktiona. Olettaen, että kiertoaika pysyy suurimman mahdollisen fyysisen puuvolyymien tuottamaa kiertoaikaa lyhempänä, *in situ* arvostukset lisäävät puun tarjontaa ja alentavat puun markkinahintaa.

Usean puuston mallissa tarkasteltiin tapausta, jossa kunkin puuston *in situ* merkitys on sidoksissa koko metsän ikäluokkarakenteeseen. Numeerisen simuloinnin perusteella *in situ* arvostusten seurauksena syntyy taloudellinen kannustin lisätä metsän ikäluokkarakenteen heterogeenisuutta eli metsää sopeutetaan alkuperäisestä ikäluokkarakenteesta hitaasti kohti normaalimetsää. Vastaavaa taloudellista kannustinta ei synny ilman *in situ* arvostuksia olettaen, että pääomamarkkinat ovat täydelliset.

Myös metsämaan markkina-arvon määräytyminen eroaa oleellisesti klassisista Faustmannin tuloksista. Markkina-arvo riippuu mm. metsänomistajan varallisuudesta ja *in situ* hyötyihin liittyvistä omistusoikeuksista. Jokamiehenoikeuksien tapauksessa potentiaaliselle ostajalle syntyy taloudellinen kannustin antaa liioiteltu kuva *in situ* tekijöiden vaikutuksesta soveltamaansa metsänhoitoon. Metsänomistajan avainbiotoopin säästämisestä (aidosti) vaatima korvaus riippuu esimerkiksi hänen tulotasostaan (vrt. uusi metsälaki) ja metsänomistajalle syntyy kannustin vähätellä omia ei-puuntuotannollisia arvostuksiaan. Mallin perusteella voidaan esittää hypoteesi, jonka mukaan jokamiehenoikeuksien tapauksessa metsänomistajien *in situ* arvostuksilla on pienempi alentava vaikutus metsän puuntuotannolliseen arvoon kuin yksityisomistuksen tapauksessa.

Key words: forest amenity values, forest rotation, timber supply

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# 1 Introduction

Recent developments in forest economics and timber supply models have led to two distinct traditions. These are the two-period or Scandinavian tradition and the optimal rotation or North American tradition (Amacher 1997). The former is based on the Fisherian utility maximization consumption/savings model augmented by a biomass harvesting model. The optimal rotation tradition is based on the Faustmann (1849) profit maximization model where harvesting activity is related to stands with specific ages. The two period tradition has been successful in incorporating varying factors determining timber supply, such as uncertainty in prices and interest rates, nontimber values, imperfect capital markets and forest taxation (Lohmander 1983, Johansson and Löfgren 1985, p. 268, Koskela 1989, Kuuluvainen 1990, Ovaskainen 1992, Ollikainen 1993, Amacher and Brazee 1997). A central feature in the two-period tradition is that forest harvesting decisions become dependent on forest owner-specific characteristics, a hypothesis that has obtained clear empirical support (Binkley 1981, Kuuluvainen and Salo 1991, Dennis 1990). In contrast, the tradition based on the Faustmann rotation model continues to implicitly rely on the Fisherian separation theorem, even in models with nontimber values where this simplification can be ruled out by the results of the two-period tradition. However, as a forest model the Faustmann specification is superior since it is based on stands with specific ages, a feature that is directly related to coniferous trees forest management<sup>1</sup>. Thus there is a need to combine the strengths of the two traditions and formulate a utility-based rotation model that is able to explain the relationship between forest owner characteristics and timber supply in terms of the optimal rotation period.

We propose is a utility-based infinite horizon rotation model with nontimber

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<sup>1</sup>It is shown by Binkley (1987) that the "lumbered" parameter biomass approach and the rotation model are based on different biotechnologies and lead to fundamentally different optimality conditions. See also Johansson and Löfgren (1985) p. 8 and 55.

values and with the forest owner's consumption/savings decision making. The forest owner lives infinitely or has an altruistic bequest motive in the sense that his bequest function equals the indirect utility function of his heirs. Capital markets are perfect and the owner obtains nonforest income. *In situ* benefits depend on stand age. The resulting nonindustrial private forest owner's optimal forest harvesting has several properties that surprisingly differ from the results in both the two-period and optimal rotation traditions.

We show that if the forest owner's subjective time preference equals the rate of interest, the rate of consumption is constant through time and it may be optimal to apply a constant rotation period forever. In such a case all comparative static properties of the rotation period (and timber supply) differ from both the Faustmann (1849) and Hartman (1976) models. In addition, the value of forest land becomes dependent on forest owner characteristics and the property rights related to nontimber values of forests. The initial stand age determines the owner's wealth and hence also the length of the optimal rotation period. Similar dependence is ruled out in the Faustmann (1849) and Hartman (1976) models. Simultaneously with the Faustmann -type stationary harvesting program, there may exist other optimality candidates with nonconstant rotation period and finite number of rotations. With "high" nontimber values, such a solution may be the global optimum. This is one implication of the fact that with consumption/savings dynamics the forest owner may not face a chain of identical harvesting problems.

A forest owner with increasing consumption and accumulating financial assets always applies nonconstant rotation and gives up harvesting in finite time. By contrast, an owner with decreasing assets shortens the rotation through time and rotation converges toward the Faustmann solution. Recall that in contrast the Faustmann (1849) model and its one-stand (autonomous) extensions yield constant rotation that is repeated forever. In a case with multiple stands we show that the harvests of different stands are linked together both via the budget constraint and because the *in situ* value of any stand may depend on the age

structure of the whole forest. We demonstrate that *in situ* valuation may yield a development toward forests with increasing heterogeneity in year classes and toward harvesting in smaller units.

All the above results are new. Our model differs from multiple use models like Hartman (1976) or Strang (1983) in the optimal rotation tradition since we include the forest owner's consumption/savings decision making. Because we consider the optimal rotation period, our model differs from any model in the two-period tradition based on a biomass description of forests. Our model and results also differ from Hyberg and Holthausen (1989) since their formulation does not fully capture the intertemporal nature of the problem and is restricted to the single stand case. Tahvonen (1997) develops a utility-based rotation model with consumption/savings dynamics and imperfect capital markets, but the *in situ* version of the model considers a forest owner without a bequest motive, i.e. it is restricted to the single harvest case. Bowes and Krutilla (1985) studies a multiple stand model for public forestry planning. In contrast to their model, our formulation describes a private nonindustrial forest owner, including his consumption/savings planning, and shows how the problem can be solved by nonlinear programming instead of Bellmann's dynamic programming.

Section 2 specifies the problem and shows how the optimal control problem with endogenous stock discontinuities can be solved by combining a simple optimal control and a nonlinear programming problem. Section 3 studies the case of constant consumption, its comparative statics and the value of forest land. Section 4 considers evolving consumption and non-stationary rotation. Section 5 extends the model to the multiple stand case. In section 6 we discuss possible future extensions.

## 2. Even-aged forest harvesting and optimal consumption/savings decisions

### *Problem specification and notation*

We define a stand as a smallest harvestable unit of even-aged trees. Without restrictions on the smallest harvestable unit, one may obtain outcomes where a forest of any size is converted to a normal forest. Physically, the smallest harvestable unit is one tree, but it may be reasonable to assume that the stand consists some larger number of trees since harvesting trees in very small units may lead to high harvesting unit costs. In addition, with coniferous trees, small stand size reduces growth since stand size determines lighting conditions. In some parts of Scandinavia this has lead to legislation that practically rules out stand size so small as to lead to uneven-aged management.

The concept of an even-aged stand leads us to a formulation that accords with multiple use single stand models like Hartman (1976) and others. In our multiple stand version, the harvestable land area may consists of any number of stands of varying ages. We first develop analytical results for a single stand model and then generalize it to consist of any number of stands.

Assume that a forest owner has one even-aged stand. The stand volume is denoted by  $x$  ( $m^3$ ), with  $\dot{x}=F(x)$ , where  $F$  is a strictly concave growth function such that  $F(0)=F(\bar{x})=0$  and  $\bar{x}>0$ . Denote stumpage price by  $p$  and planting costs by  $w$ . At harvesting moments, the owner immediately plants the next age class and thus obtains net forest income equal to  $px(s_i)-w$ , where  $x(s_i)$  denotes the harvesting volume for a stand of age  $s_i$ . Subscript  $i$  refers to the number of the harvest and  $i=1,\dots,N$ ,  $N\leq\infty$ . The net forest income is added to the owner's financial assets, denoted by  $a(t)$ , where  $t$  is calendar time. The owner receives nonforest income  $m(t)$  and interest on his savings. The rate of interest is  $\rho$ , the subjective time preference is  $\delta$ . Capital markets are perfect. Financial assets are

decreased by consumption expenditures  $c(t)$ . Instantaneous utility from consumption is given by a strictly concave function  $U(c)$ . In addition, the stand of age  $s$  gives the owner instantaneous *in situ* utility  $A(s)$ . We assume that  $A(s)$  is nondecreasing and concave, not because this would be the only empirically plausible form but in order to obtain sound results for a clearly defined subclass of possible *in situ* valuation functions.<sup>2</sup>

In the Faustmann model, infinite horizon reflects perfectly functioning land markets. As will be shown, the inclusion of *in situ* preferences complicates the role of land markets. We derive the optimal forest rotation by assuming that the forest owner has a perfectly altruistic bequest motive. The bequest function is assumed to equal the indirect utility function of the next human generation with identical preferences. If all generations have the same preferences, all financial assets and consumption remains continuous when the bequest is realized. Such bequest motives need not necessarily be considered explicitly. We obtain the same outcome by assuming that the forest owner lives forever. The maximized utility for this solution then forms the basis for a possible decision to sell the forest land.

Denote the volume of the stand just after planting by  $x_0$ . The forest owner then chooses the consumption schedule  $c(t)$  and the cutting moments  $t_i$ ,  $i=1, \dots, N$  so as to

$$\max_{\{t_i, c\}} W = \int_0^{\infty} [U(c) + A(s)] e^{-\delta t} dt \quad (1)$$

$$\text{s.t.} \quad \dot{a} = \rho a - c + m, \text{ when } t \neq t_i, \quad (2)$$

$$\dot{x} = F(x), \text{ when } t \neq t_i, \quad (3)$$

$$a(t_i) = a(t_i^-) + p x(t_i^- - t_{i-1}) - w, \quad (4)$$

$$x(t_i) = x_0, \quad (5)$$

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<sup>2</sup>Several papers assume that nontimber benefits are a function of stand volume. This may be problematic because not all forestry practices that increase stand volume increase the *in situ* value of forests.

$$a(0)=a_0, \quad (6)$$

$$x(0)=x_0, \quad (7)$$

$$\lim_{t \rightarrow \infty} a(t) \geq 0, \quad (8)$$

where  $t_i^-$  denotes the left hand limit for  $t_i$ . Problem (1)-(8) includes two time variables: the calendar time  $t$  and the age of the stand  $s$ . The age of the stand equals the calendar time minus the date of last harvest, i.e.  $s(t)=t-t_i$ , where  $t \in (t_i, t_{i+1}]$ ,  $t_0=0$ . To simplify the notation, the first planting is assumed to take place at the initial date  $t_0=0$ . In addition, we write  $t_{N+1}=\infty$  if  $N<\infty$ . At the harvesting moments  $t_i$ , stand volume jumps to zero and capital assets jump from  $a(t_i^-)$  to  $a(t_i^-)+px(s_i)-w$ . This discontinuity in states complicates the problem. We present two different approaches for developing the necessary optimality conditions. Appendix 1 applies the maximum principle, and here we develop identical conditions using a two step method that more intuitively reveals the structure of the problem.

#### *Partial separation between forestry and consumption decisions*

Applying conditions (2), (4), (6) and (8) we can write the forest owner's budget constraint as:

$$B=a_0+\int_0^{\infty} e^{-\rho t} m(t) dt + \sum_{i=1}^N [px(t_i-t_{i-1})-w] e^{-\rho t_i} - \int_0^{\infty} e^{-\rho t} c(t) dt \geq 0. \quad (9)$$

Due to perfect capital markets, the present values of future incomes determine the consumption possibilities but not their timing along the planning horizon. Given any harvesting program  $s_1, \dots, s_N$ , the forest owner's problem is to use the budget  $b_0=$

$a_0+\int_0^{\infty} e^{-\rho t} m(t) dt + \sum_{i=1}^N [px(t_i-t_{i-1})-w] e^{-\rho t_i}$  so as to

$$\max_{\{c\}} \int_0^{\infty} U(c) e^{-\delta t} dt, \text{ s.t. } \dot{b}=\rho b-c, b(0)=b_0, \lim_{t \rightarrow \infty} b(t) \geq 0. \quad (10)$$

The solution problem satisfies  $\lambda(t)=\lambda_0 e^{(\delta-\rho)t}$  and  $b_0=\int_0^\infty e^{-\rho t} c(t,\lambda_0) dt$ , where  $c(t,\lambda_0)$  is defined by  $U'[c(t)]=\lambda(t)$ . Using this solution we write the forest owners problem as

$$\max_{\{\lambda_0, t_1, \dots, t_N\}} W = \int_0^\infty U[c(t,\lambda_0)] e^{-\delta t} dt + \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} A(t-t_{i-1}) e^{-\delta t} dt \quad (11)$$

subject to (9), where  $c=c(t,\lambda_0)$ . This leads to the Lagrangian  $L=W+\mu B$  and to the necessary conditions

$$\partial L / \partial \lambda_0 = \int_0^\infty e^{-\delta t} U'(c) \partial c / \partial \lambda_0 dt - \mu \int_0^\infty e^{-\rho t} \partial c / \partial \lambda_0 dt = (\lambda_0 - \mu) \int_0^\infty e^{-\rho t} \partial c / \partial \lambda_0 dt = 0, \Rightarrow \lambda_0 = \mu, \quad (12)$$

$$\begin{aligned} \partial L / \partial t_i \Big|_{\mu=\lambda_0} = & \lambda_0 e^{-\rho t_i} \left\{ pF[x(s_i)] - pF[x(s_{i+1})] e^{-\rho s_{i+1} - \rho} [px(s_i) - w] \right\} + \\ & e^{-\delta t_i} \left\{ A(s_i) - A(0) - \int_0^{s_i+1} A'(s) ds \right\} = 0, \end{aligned} \quad (13)$$

together with (9) as an equality. Note in (12) that  $U'(c)=\lambda_0 e^{(\delta-\rho)t}$ . Equation (13) together with  $\lambda_0$  given by the solution to problem (10) form the basis for studying the optimal harvest decisions.

It is useful to first derive the Fisherian separation result and the Faustmann formula as special cases. Assuming  $A(s)=0$ , equation (13) simplifies to

$$pF[x(s_i)] - pF[x(s_{i+1})] e^{-\rho s_{i+1} - \rho} [px(s_i) - w] = 0. \quad (14)$$

Equation (14) links consecutive rotation periods  $s_i$  and  $s_{i+1}$  together. The solution for (14) is independent of the marginal value of consumption, implying that the optimal harvesting program must be independent of forest owner characteristics like income and preferences. It

is also independent of the subjective time preference and calendar time. Thus, we can follow Faustmann (1849, p. 442) or Clark (1990, p. 269) and conclude that all rotation periods must have equal length, i.e.  $s_i = s_{i+1}$ ,  $i=1, \dots, \infty$ . With equal rotation length (14), yields the Faustmann rotation. Another possibility is to view equation (14) as a nonlinear, first order, autonomous ordinary difference equation written in implicit form. It has the Faustmann rotation  $s_i = s_f$  for  $i=1, \dots, \infty$  as a stationary solution. This solution is unstable because  $ds_{i+1}/ds_i \Big|_{s_i=s_f} = e^{\rho s_f} > 1$ . Any other solution with  $N=\infty$  can be shown to violate the second order necessary conditions. Thus the Faustmann solution must be the global optimum.

We showed that without *in situ* preferences our model yields the result that forest harvesting decisions are made independently of consumptions/savings decisions (c.f. the Fisherian separation theorem) and that the stand is harvested according to the Faustmann solution. The notion that an optimal rotation program can be viewed as a solution to a nonlinear difference equation will turn out to be highly useful when we consider more complex specifications.

### 3. Optimal rotation with constant consumption

#### *Stationary rotations*

The Faustmann solution was obtained independently of whether  $\delta \gtrless \rho$ . When the forest is valued *in situ*, the difference between the rate of discount and the subjective time preference becomes essential. In this section we study the case where  $\delta = \rho$ . This implies that the costate for capital assets is constant, and we write  $\lambda = \lambda_0$  and  $c = c_0$ . Given any constant  $\lambda$  we can view equation (13) as a first order, autonomous difference equation. We first study a Faustmann -type stationary optimality candidate where the rotation length is a finite constant;  $s_i = s_1$  for  $i=2, \dots, \infty$ . By partial integration,  $A(s_1) - A(0) - \int_0^{s_1} A'(s) e^{-\delta s} ds = A(s_1)(1 - e^{-\delta s_1}) -$

$\delta \int_0^{s_1} A(s)e^{-\delta s} ds$ . Using this and dividing (13) by  $(1-e^{-\delta s_1})$  yields the cutting equation in the form

$$A(s_1) + \lambda_0 pF[x(s_1)] = \lambda_0 \delta px(s_1) + \delta \lambda_0 [px(s_1)e^{-\delta s_1 - w}] / (1 - e^{-\delta s_1}) + \delta \int_0^{s_1} A(s)e^{-\delta s} ds / (1 - e^{-\delta s_1}). \quad (15)$$

The LHS of (15) equals the *in situ* benefits and value growth that can be obtained by marginally postponing the cutting moment. The RHS equals the utility costs due to delaying the income from the next cut  $\lambda_0 \delta px(s_1)$ , plus the monetary loss due to delaying the whole future harvesting schedule, plus the loss in delaying the future stream of *in situ* benefits. Note that all terms are evaluated in utility units. The Faustmann formula follows as a special case of (15) when  $A(s) \equiv 0$  and the Hartman (1976) formula when  $\lambda_0 \equiv 1$ . Equation (15) can be written in the form<sup>3</sup>

$$\Gamma \equiv \lambda_0 \left\{ pF[x(s_1)] - \delta px(s_1) - \delta [px(s_1)e^{-\delta s_1 - w}] / (1 - e^{-\delta s_1}) \right\} + A(s_1) - \delta \int_0^{s_1} A(s)e^{-\delta s} ds / (1 - e^{-\delta s_1}) = 0. \quad (16)$$

In the Faustmann solution  $G \equiv \left\{ pF[x(s_1)] - \delta px(s_1) - \delta [px(s_1)e^{-\delta s_1 - w}] / (1 - e^{-\delta s_1}) \right\} = 0$  where it must hold that  $\partial G / \partial s_1 \big|_{G=0} = pF[x(s_1)] \{ F'[x(s_1)] - \delta \} < 0$ . Such a maximum must be unique because the concavity of  $F$  rules out the change in sign of  $\partial G / \partial s_1$  at points when  $G=0$ . Because  $A(s_1) - \delta \int_0^{s_1} A(s)e^{-\delta s} dt / (1 - e^{-\delta s_1}) > 0$  we obtain the result that  $G < 0$  and that *in situ* benefits lengthen the rotation period.

Assume that nonforest income  $m$  is a constant. From the budget constraint (9):

$$c_0 = m + \delta a_0 + \delta [px(s_1) - w] e^{-\delta s_1} / (1 - e^{-\delta s_1}). \quad (17)$$

<sup>3</sup>Compare the Faustmann formula (4.4) in Johansson and Löfgren (1985, p. 80).

Thus consumption equals nonforest income plus the interest on the initial assets and on the (monetary) value of planted forest land.

By the uniqueness of the Faustmann solution,  $G < 0$  for  $s \in (s_f, \infty)$ . It also remains bounded from below and  $G \rightarrow \delta w - \delta p x(\infty)$  as  $s \rightarrow \infty$ . The term  $A(s_1) - \delta \int_0^{s_1} A(s) e^{-\delta s} dt / (1 - e^{-\delta s_1})$  remains strictly positive for any  $s \in [s_f, \infty)$ . Note from (17), that  $\partial c_0 / \partial a_0 = \delta > 0$  implying by the strict concavity of  $U$  that  $\partial \lambda_0 / \partial a_0 < 0$ . Thus if we let  $a_0$  increase enough, the LHS of (16) must finally obtain positive values for any finite rotation period. This implies that with  $a_0$  high enough there cannot be a stationary rotation program satisfying (16).

To study the other cases, note that always when  $\lim_{s_1 \rightarrow \infty} \Gamma(s_1) = -\lambda_0 \delta [p x(\infty) - w] + A(\infty) - \delta \int_0^\infty A(s) e^{-\delta s} dt$  is positive there is one local maximum with an infinitely long rotation period. This is the case e.g. when  $A(s)$  is linear. If there is simultaneously some finite locally optimal  $s_1$ , the globally optimal solution is the candidate with highest  $W$ , even allowing for the possibility that the stand is not planted at all. If  $\lim_{s_1 \rightarrow \infty} \Gamma(s_1)$  is negative, the globally optimal solution is the finite horizon candidate that yields the highest value for  $W$ . With general functional forms, it seems difficult to rule out the multiple interior optimality candidates. The global optimum is then the candidate that yields the highest  $W$ . The fact that *in situ* benefits lead to multiple local optima holds even in the Hartman (1976) model with concave  $A(s)$  (see e.g. Strang 1983).

### *Nonstationary solutions*

We can study the stability of the stationary rotation as in the case of the Faustmann solution. For  $\delta = \rho$  we can study the stability of (13) by keeping  $\lambda_0$  as a constant. Differentiating (13) yields  $ds_{i+1} / ds_i \Big|_{s_i = s_{i+1}} = e^{\delta s_i} > 1$ , implying that the stationary solution is unstable in the sense that there cannot be asymptotic convergence toward it. In the Faustmann model and Hartman (1976) it is always optimal either not to harvest at all or to harvest infinitely. This follows because after any harvest the optimization problem is

repeated identically. In our model we cannot rely on this simple argument because even with constant consumption it is possible that financial assets accumulate, implying that the forest owner's situation need not be the same after each harvest. To study whether it may be optimal to apply some finite number of harvests, consider Table 1.

Table 1. Optimal number of rotations

N	0	1	2	3*	4	5	$\infty$
$W_1$	8877.20	9225.63	9279.46	9286.56	9286.06	9284.93	9283.35
$s_1$		36.235	35.845	35.862	35.917	35.955	35.997
$s_2$			36.000	35.871	35.906	35.946	$\infty$
$s_3$				35.890	35.884	35.927	
$s_4$					35.838	35.890	
$s_5$						35.813	

Note:  $A=As$ ,  $(U=c^{1-\alpha}-1)/(1-\alpha)$ ,  $x=K/(1-Ce^{-rt})$ ,  $C=(x_0-K)/x_0$ ,  $A=1.4$ ,  $\alpha=0.5$ ,  $\rho=\delta=0.02$ ,  $K=500$ ,  $r=0.15$ ,  $x_0=10$ ,  $p=500$ ,  $w=2000$ ,  $a_0=10^5$ ,  $m=1000$ .

Comparing the no harvesting solution ( $N=0$ ) and infinite stationary rotations ( $N=\infty$ ) yields the result that it is preferable to apply a rotation length of 35.997 infinitely. However, computing rotations for a finite number of periods yields the result that the present value utility increases up to three rotations and decline thereafter. Thus the globally optimal solution may not be found by comparing only the stationary infinite rotation program and the no harvesting solution.

With a finite number of rotations,  $\lambda_0$  must equal the marginal utility of constant consumption that corresponds to the level of accumulated financial assets after the last harvest. It is clear that the level of financial assets must be higher after the last harvest than at  $t=0$  because otherwise applying the no harvesting policy from  $t=0$  would yield both higher consumption and higher *in situ* benefits. Note that after the last harvest it is necessary that  $\lambda_0\{pF[x(s_{N+1})]-\delta[px(s_{N+1})-w]\}+A(s_{N+1})-\int_0^\infty A'(s)e^{-\delta s}ds>0$  as  $s_{N+1}\rightarrow\infty$  (cf. equation 16). Otherwise stopping the harvesting cannot be optimal. This condition is the

more likely to hold the lower the value of  $\lambda_0$ . The Faustmann rotation program yields the lowest attainable value for  $\lambda_0$ . Thus, if the above condition is violated when  $\lambda_0$  corresponds to the Faustmann level, it follows that the infinite stationary rotation program is the globally optimal solution. It is easy to develop such examples (e.g. when  $A \leq 1.2$  in the example of Table 1).

The numerical computation of rotation programs with any finite  $N$  can be carried out by solving

$$\max W(t_1, \dots, t_N) = U[c(t_1, \dots, t_N)] / \delta + \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} e^{-\rho t} A(t-t_{i-1}) dt \quad (18)$$

where  $c(t_1, \dots, t_N)$  is defined by (9) as an equality. This is a nonlinear unconstrained optimization problem. The global optimum is found by picking the level of  $N$  that yields the highest value for  $W$ .

The property of a finite rotation program in combination with nonconstant rotation differs sharply from the previous results in the multiple use literature. It is one implication of the fact that with consumption/savings dynamics the forest owner may not face a chain of identical harvesting problems. These properties become even more evident when we consider cases where  $\delta \neq \rho$ .

#### *Comparative statics for the stationary rotation program*

We next study how the stationary rotation depends on the model parameters. Note from (11) and (17) that in the case of constant infinitely lasting rotations the optimal rotation problem can be formulated as

$$\max_{\{s_1\}} W(s_1) = U[c_0(s_1)] + \int_0^{s_1} A(s_1) e^{-\delta s} ds / (1 - e^{-\delta s_1}),$$

where  $c_0(s_1)$  is defined by (17). This is a one -variable unconstrained optimization problem. Carrying out the differentiation, one obtains (16) as the first order necessary condition. Thus for any local interior maximum it must hold that  $\partial\Gamma/\partial s_1 < 0$  from equation (16). We obtain

$$\partial\Gamma/\partial m = \partial\lambda_0/\partial m G > 0, \Rightarrow ds_1/dm > 0, \quad (19)$$

$$\partial\Gamma/\partial a_0 = \partial\lambda_0/\partial a_0 G > 0, \Rightarrow ds_1/da_0 > 0, \quad (20)$$

$$\partial\Gamma/\partial p = \partial\lambda_0/\partial p G + \lambda_0 \{ F[x(s_1)] - \delta x(s_1) - \delta x(s_1) e^{-\delta s_1/(1-e^{-\delta s_1})} \} \geq 0, \Rightarrow ds_1/dp \geq 0 \quad (21)$$

$$\partial\Gamma/\partial w = \partial\lambda_0/\partial w G + \lambda_0 \delta / (1 - e^{-\delta s_1}) \geq 0, \Rightarrow ds_1/dw \geq 0 \quad (22)$$

$$\partial\Gamma/\partial \delta = \partial\lambda_0/\partial \delta G + \lambda_0 \left\{ \delta \int_0^{s_1} A(s) e^{-\delta s} ds / (1 - e^{-\delta s_1}) \right\} / \partial \delta \geq 0, \Rightarrow ds_1/d\delta \geq 0, \quad (23)$$

where  $\partial\lambda_0/\partial c_0 < 0$  and by (17) it holds that  $\partial c_0/\partial m = 1 > 0$ ,  $\partial c_0/\partial a_0 = \delta > 0$ ,  $\partial c_0/\partial p = \delta x(s_1) e^{-\delta s_1/(1-e^{-\delta s_1})} > 0$ ,  $\partial c_0/\partial w = -\delta e^{-\delta s_1/(1-e^{-\delta s_1})} < 0$ ,  $\partial c_0/\partial \delta = e^{-\delta s_1} [px(s_1) - w] (1 - \delta s_1 - e^{-\delta s_1}) / (1 - e^{-\delta s_1})^2 + a_0 \geq 0$ .

Nonforest income and initial nonforest assets lengthen the rotation period. Recall that these effects are totally absent from the Faustmann (1849) and Hartman (1976) formulations. The effects of price and regeneration costs are ambiguous because there are both income and substitution effects. These results are unambiguously negative and positive in both the Faustmann and Hartman models (Bowes and Krutilla 1985). The Faustmann rotation depends only on the relative values of  $p$  and  $w$ . In our model, the absolute values of  $p$  and  $w$  also affect the rotation period. The effect of the rate of interest/subjective time preference is ambiguous, as in the Hartman (1976) model.

In the Faustmann and Hartman (1976) models, the initial stand age does not affect the optimum rotation period. Above, we have assumed that at  $t=0$  the forest owner has a newly planted forest land. It is possible to consider any initial stand age, with the implication that the older the initial stand the longer the rotation period. This follows

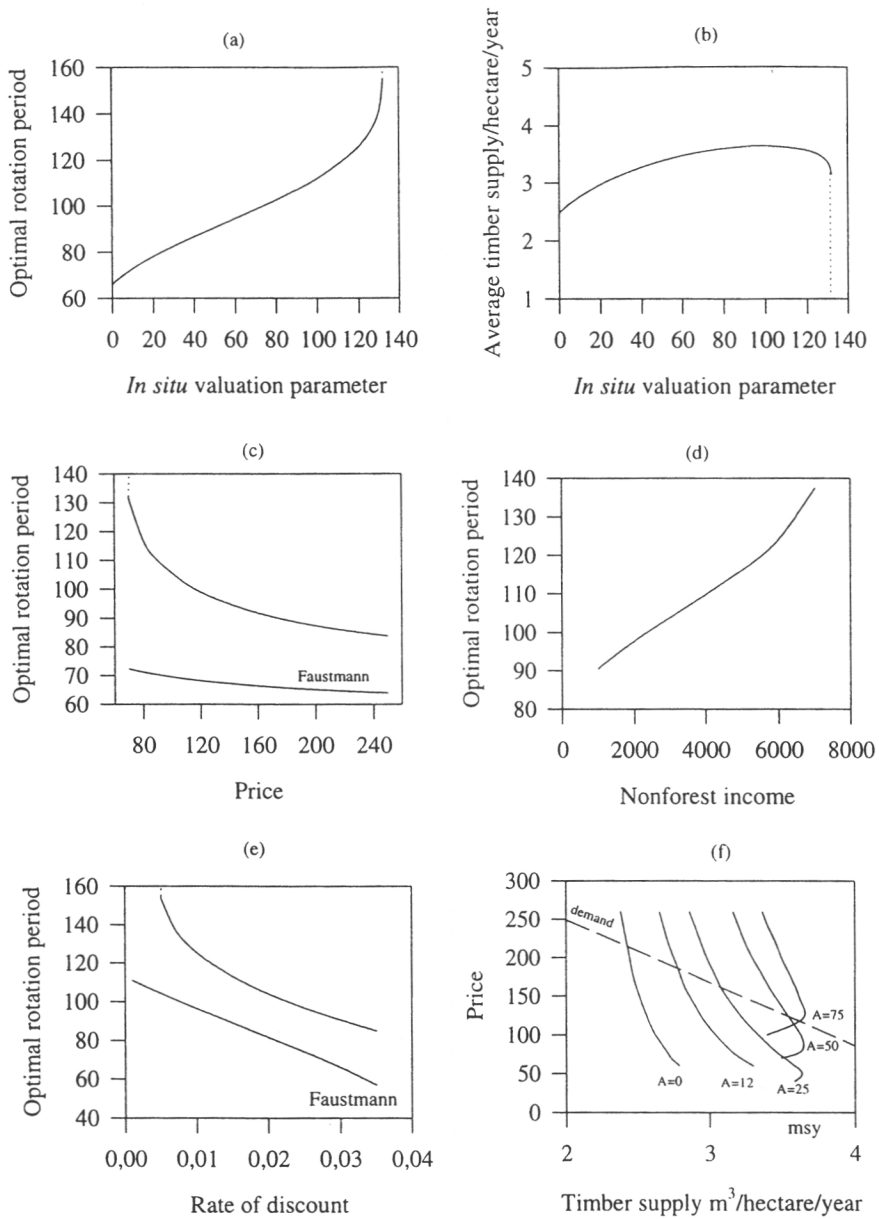


Figure 1. Comparative statics

Note:  $A(s)=A(1-e^{-ks})$ ,  $U(c)=(c^{1-\alpha}-1)/(1-\alpha)$ ,  $x(s)=K/(1-Ce^{-rs})$ ,  $C=(x_0-K)/x_0$ ,  $A=35$ ,  $k=0.01$ ,  $x_0=10$ ,  $K=500$ ,  $r=0.048$ ,  $p=170$ ,  $w=2000$ ,  $m=1000$ ,  $\delta=p=0.03$ ,  $a_0=1000$ ,  $\alpha=1/2$ ,  $A=50$ , if not stated otherwise.

because with older initial stand the forest owner has more valuable total assets and more financial resources to extend the rotation and benefit from the value of the stand *in situ*.

Figure 1 shows numerical examples for the stationary rotation period. In Fig. 1a the *in situ* valuation increases the rotation length from the Faustmann solution ( $\approx 66$ ) and the rotation length finally becomes infinitely long. Fig. 1b shows that the annual average timber supply increases with increasing *in situ* valuation until the maximum sustainable yield ( $\approx 3.8\text{m}^3$ ) is reached. Fig. 1c demonstrates that with *in situ* valuation the rotation period may become more sensitive to price changes as compared to the Faustmann solution. Fig. 1d shows the dependence of optimal rotation on nonforest income. Fig. 1e shows a case where an increase in the rate of interest/subjective time preference decreases both rotation periods. Fig. 1f shows the backward bending supply curves. *In situ* valuation shifts these curves to the right implying that a marginal increase in *in situ* valuation increases timber supply and decreases price, given that the rotation period is shorter than the maximum sustainable yield rotation and *vice versa*.

#### *Value of forest land*

One main aim in the famous Faustmann (1849) paper was to obtain a method for calculating the value of forest land<sup>4</sup>. In his model the value just after regeneration equals  $[px(s_f)-w]e^{-\delta s_f}/(1-e^{-\delta s_f})$ , and the value is independent of forest owner characteristics. This does not hold when forests are valued *in situ*. We study the value of forest land in the case of infinite stationary rotations. Assume first a private property regime in which the forest owner obtains *in situ* value from the forest only if it is in his possession. In this case the forest owner is ready to sell the forest land if the price offer exceeds  $V_1$  defined by

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<sup>4</sup>"The practical importance of this calculation is easy to see. From it we obtain the necessary information on the forest value in such cases as voluntary and enforced sales, destruction of the forest by fire, insects, man, etc., and assesment of the most advantageous system and length of rotations." (Faustmann 1849, p. 442)

$$U(c_0)/\delta + \int_0^{s_1} A(s)e^{-\delta s} ds / (1 - e^{-\delta s_1}) - U[\delta(a_0 + V_1) + m] / \delta = 0, \quad (24)$$

where  $c_0$  is given by (17). Note that without the forest land the present value maximized utility equals  $U(\delta a_0 + m) / \delta$ . Obviously  $V_1$  must exceed the present monetary value of the rotation program of the original owner, but it must also exceed the Faustmann value of the forest land. Differentiation shows<sup>5</sup> that  $V_1$  is higher the higher the timber price, *in situ* valuation, nonforest income and nonforest assets and the lower the regeneration costs. The effect of the rate of interest or subjective time preference is ambiguous. Among other things this shows that buyers with higher nonforest income or nonforest assets, *ceteris paribus*, are willing to buy forest stands from less wealthy owners. This follows because with high nonforest income the marginal utility from consumption is low.

Assume next that the stand *in situ* is a common property resource (c.f. the Right of Common Access applied in Scandinavia) In this case the forest owner will sell his forest land if the offer exceeds  $V_2$  as defined by

$$U(c_0)/\delta + \int_0^{s_1} A(s)e^{-\delta s} ds / (1 - e^{-\delta s_1}) - \{U[\delta(a_0 + V_2) + m] / \delta + \int_0^{\underline{s}} A(s)e^{-\delta s} ds / (1 - e^{-\delta \underline{s}})\} = 0, \quad (25)$$

where  $\underline{s}$  is the expected rotation of the new owner of the forest land. When the *in situ* value of the forest is a common property resource it is natural to assume that any potential buyer obtains *in situ* utility even before a transaction. If  $\underline{s} < s_1$ , the seller requires that  $V_2$  exceed the monetary value of the rotation related to  $s_1$ . Otherwise he would be worse off. However,  $V_2$  must also exceed the monetary value related to rotation  $\underline{s}$  because the seller could have applied  $\underline{s}$  but chose instead  $s_1$  as the global maximum. Thus if  $\underline{s} < s_1$ , it is necessary that  $V_2$  exceeds the monetary value related to  $\underline{s}$  for a transaction to be beneficial to the seller.

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<sup>5</sup>Use the envelope theorem and the fact that  $U'(c_0) > U'(\delta a_0 + \delta V_1 + m)$ .

However, given that the buyer's *in situ* valuation function is increasing with rotation length, a transaction where  $V_2$  exceeds the monetary value related to  $\underline{s}$  and  $\underline{s} < s_1$  cannot be beneficial for the buyer because he will be worse off both in monetary terms and *in situ* utility. It is thus necessary that the buyer is planning to apply at least as long a rotation period as the present owner, up until a transaction is carried out. When  $\underline{s} > s_1$  the seller's *in situ* benefits increase, implying that  $V_2$  is lower than the monetary benefits related to  $s_1$ . Differentiation of (30) shows<sup>6</sup> that now  $V_2$  is the lower, the higher the values of  $m$  and  $a_0$  for the present owner and the higher his *in situ* valuation. The effects of  $p$  and the rate of discount are ambiguous. It is also clear that in the common property regime a potential buyer has an incentive to give the impression that he will apply a longer rotation than what he is actually planning.

In the new Finnish forest law, forest owners are compensated if they apply extended rotations to preserve the most valuable parts of their forests. We study how much the owner must be compensated until he is ready to totally give up forest harvesting. Such compensation must exceed  $V_3$  defined by

$$U(c_0)/\delta + \int_0^{s_1} A(s)e^{-\delta s} ds / (1 - e^{-\delta s_1}) - U(\delta a_0 + m + \delta V_3) / \delta - \int_0^{\infty} A(s)e^{-\delta s} ds = 0. \quad (26)$$

Differentiation shows that  $V_3$  is the higher, the less wealthy the forest owner, the lower his *in situ* valuation and the higher the timber price. In this case the forest owner has an incentive to inform the officials of shorter rotations than those consistent with his real preferences.

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<sup>6</sup>Again using the envelope theorem and the fact that  $U'(c_0) < U'(\delta a_0 + m + \delta V_2)$ .

## 4. Nonstationary rotations and evolving consumption

We next study the case where  $\delta \neq \rho$ . Recall that the optimal harvesting program must satisfy equation (13):

$$\lambda_0 e^{(\delta-\rho)t_i} \left\{ pF[x(s_i)] - \rho [px(s_i) - w] - pF[x(s_{i+1})] e^{-\rho s_{i+1}} \right\} + A(s_i) - A(0) - \int_0^{s_{i+1}} A'(s) e^{-\delta s} ds = 0. \quad (13)$$

Define  $D \equiv pF[x(s_i)] - \rho [px(s_i) - w] - pF[x(s_{i+1})] e^{-\rho s_{i+1}}$ . Note that  $D$  is bounded. When  $\delta \neq \rho$  the rotation period cannot equal any finite constant because with constant  $s_i$ ,  $D$  and  $A(s_i) - A(0) - \int_0^{s_{i+1}} A'(s) e^{-\delta s} ds$  are constant while  $\lambda_0 e^{(\delta-\rho)t_i}$  increases or decreases. Thus the rotation period always evolves in time when  $\delta \neq \rho$ .

Let us first study the case where  $\delta - \rho < 0$ . Now  $\lambda_0 e^{(\delta-\rho)t}$  decreases and consumption increases in time, i.e. our forest owner saves more than consumes. Assume it is optimal to continue the rotation infinitely. Because  $\lambda_0 e^{(\delta-\rho)t} D \rightarrow 0$  as  $t \rightarrow \infty$ ,  $A(s_i) - A(0) - \int_0^{s_{i+1}} A'(s) e^{-\delta s} ds$  must also approach zero as  $t \rightarrow \infty$ . Because  $A(s_i) - A(0) - \int_0^{s_{i+1}} A'(s) e^{-\delta s} ds > 0$  with any  $s_i = s_{i+1}$ , either  $s_{i+1} > s_i$  for all large enough  $i$  and  $s_i \rightarrow \infty$  as  $i \rightarrow \infty$  or  $s_i \rightarrow 0$  as  $i \rightarrow \infty$ . It cannot be optimal to have  $s_i \rightarrow 0$  because the net harvesting income  $px(s_i) - w$  becomes negative. Concavity of  $A(s)$  implies that  $\int_0^{s_i} A'(s) e^{-\delta s} ds$  remains bounded. Therefore  $A(s_i) - A(0) - \int_0^{s_{i+1}} A'(s) e^{-\delta s} ds > 0$  when  $s_i$  is large enough. This contradiction implies that when  $\delta < \rho$  it cannot be optimal to continue the forest harvesting forever.

Figure 2 shows an example of such a solution. It is computed backwards from the date of last harvest,  $t_N$ . The level of financial assets  $a(t_N)$  is set at a level implying that when looking forward it is optimal to leave the stand uncut.  $a(t_N)$  determines  $\lambda_0 e^{(\delta-\rho)t_N}$ , which makes it possible to compute the rotation periods backwards using equation (13).

Consider the case where  $\delta - \rho > 0$ . Now  $\lambda_0 e^{(\delta-\rho)t}$  increases without bound. If it is optimal to once give up harvesting, equation (13) must obtain positive values when  $s_{i+1} = \infty$

and  $s_i \rightarrow \infty$ . The concavity of  $A(s)$  implies that the term  $A(s_i) - A(0) - \int_0^{s_i+1} A'(s)e^{-\delta s} ds$  cannot increase faster than linearly while  $D$  approaches  $-\rho[p\bar{x}(\infty) - w] < 0$  as  $s_i \rightarrow \infty$ . Thus  $\lambda_0 e^{(\delta-\rho)t_i} D < 0$  and it decreases exponentially without bound, implying that equation (13) gets negative values and that it cannot be optimal to give up harvesting. The rotation period cannot increase forever by a similar argument. If the rotation period converges toward some level other than the Faustmann rotation where  $D=0$  the term  $\lambda_0 e^{(\delta-\rho)t_i} D$  increases or decreases exponentially while  $A(s_i) - A(0) - \int_0^{s_i+1} A'(s)e^{-\delta s} ds$  converges toward a constant, implying a

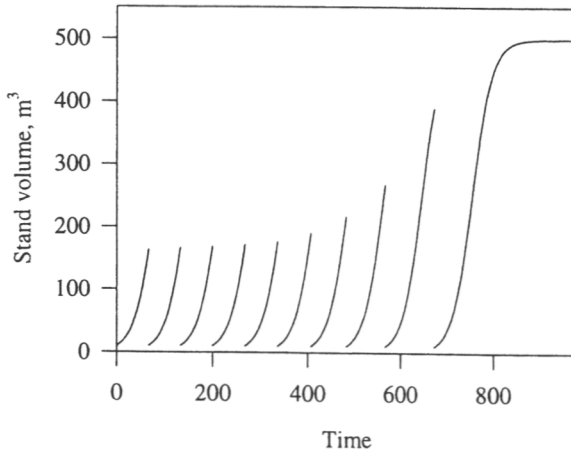


Figure 2. Optimal development of the rotation period when  $\delta < \rho$ .

Note:  $A(s) = A(1 - e^{-ks})$ ,  $U(c) = (c^{1-\alpha} - 1) / (1 - \alpha)$ ,  $x(s) = K / (1 - Ce^{-rs})$ ,  $C = (x_0 - K) / x_0$ ,  $A = 35$ ,  $k = 0.01$ ,  $x_0 = 10$ ,  $K = 500$ ,  $r = 0.048$ ,  $p = 170$ ,  $w = 2000$ ,  $m = 1000$ ,  $\delta = 0.02$ ,  $\rho = 0.03$ ,  $\alpha = 1/2$ ,  $a(t_N) = 1.13179 \cdot 10^6$

contradiction. Thus the rotation period must converge to the Faustmann rotation. Figure 3 shows a numerical example for the case where  $\delta > \rho$ . It is computed backwards by starting with a rotation period close to the Faustmann solution ( $s_f = 81$ ). However, the length of the first period is much longer and equals 117 years.

The Faustmann model and its one stand extensions usually have a constant

rotation period that is repeated forever. This is especially the case with the Hartman (1976) model and its extensions like Hyberg and Holthausen (1989).<sup>7</sup> We have shown that the rotation period need not be constant and forest harvesting may not continue forever. Thus including consumption/savings decision making in the optimal rotation model with *in situ* preferences alters some of the the most basic features of optimal forest rotation. Note also that these results cannot be obtained within the two period forest modeling tradition.

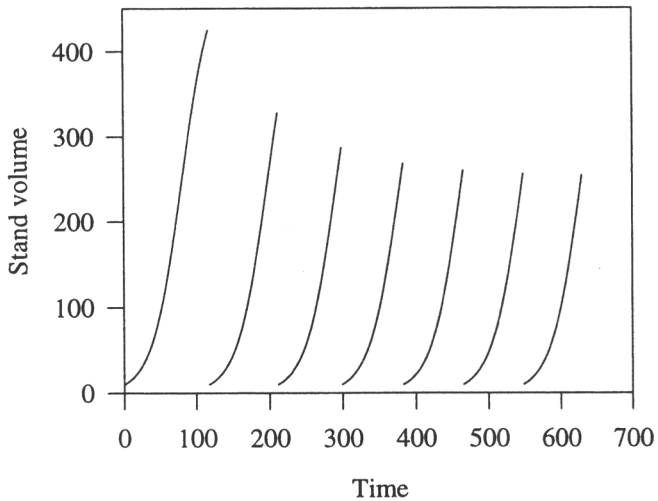


Figure 3. Optimal development of the rotation period when  $\delta > \rho$ .

Note:  $A(s) = A(1 - e^{-ks})$ ,  $U(c) = (c^{1-\alpha} - 1)/(1-\alpha)$ ,  $x(s) = K/(1 - Ce^{-rs})$ ,  $C = (K - x_0)/x_0$ ,  $A = 10$ ,  $k = 0.01$ ,  $K = 500$ ,  $r = 0.048$ ,  $x_0 = 10$ ,  $p = 170$ ,  $w = 2000$ ,  $\delta = 0.03$ ,  $\rho = 0.02$ .

<sup>7</sup>An exception is the model by Löfgren (1985), where the rotation period evolves in time because of exogenous biotechnical change.

## 5. Multiple stands

Assume that a forest consists of  $m$  stands planted at dates  $t_0^m \leq t_0^{m-1} \leq \dots \leq t_0^1 \leq 0$ . To simplify the notation we assume that  $t_0^1 = 0$ , i.e. that the youngest stand is planted at the beginning of the optimization period. The initial age distribution of the stands can thus be written as  $-t_0^k$ ,  $k=1, \dots, m$ . The *in situ* value of the forest is given by a concave symmetric function  $A(s^1, \dots, s^m)$ , where we assume that  $\partial^2 A / \partial s^i \partial s^j \neq 0$  for  $i \neq j$  and  $A \in \mathbb{C}^2$ . Thus the *in situ* value of any stand depends on the age distribution of the whole forest, but all stands make a perfectly symmetrical contribution to the forest's *in situ* value. A more general model where stands are location-specific and may have unique *in situ* value and growth functions is possible to formulate but complicates the notation.

We denote the harvesting dates for stand  $k$  by  $t_1^k, t_2^k, \dots, t_{N_k}^k$ , where the final harvest occurs at date  $t_{N_k}^k \leq \infty$ . When the number of harvests is finite, the infinite time interval after  $t_{N_k}^k$  is denoted by  $(t_{N_k}^k, t_{N_k-1}^{k-1})$  by adding one more harvest at  $t_{N_k-1}^{k-1} = \infty$ . The age of any stand  $k$  is  $s^k(t) = t - t_1^k$ , given that  $t \in [t_1^k, t_{i+1}^k)$  and its age at the  $i$ th harvesting moment is  $s_i^k = s^k(t_i^k) = t_i^k - t_{i-1}^k$ . Given any harvesting program, the present value of *in situ* utility equals  $V = \int_0^\infty e^{-\delta t} A[s^1(t), \dots, s^m(t)] dt$ . At any cutting moment, the *in situ* flow changes discontinuously. For differentiating  $V$ , we express it as a sum over time periods, where  $A[s^1(t), \dots, s^m(t)]$  is continuous. Thus we write

$$V = \int_0^\infty e^{-\delta t} A[s^1(t), \dots, s^m(t)] dt = \sum_{k=1}^m \sum_{i=1}^{N_k} \int_{t_i^{k+1}}^{t_i^k} e^{-\delta t} A(s^1, \dots, s^m) dt, \quad (27)$$

where we denote  $t_i^{m+1} = t_{i-1}^1$  in the cases of the initially oldest stand  $m$ . Note that (27) does not give the time intervals in calendar order, which does not however change the value of  $V$ . The budget constraint takes the form

$$B = a_0 + \int_0^{\infty} e^{-\rho t} [m(t) - c(t, \lambda_0)] dt + \sum_{k=1}^m \sum_{i=1}^{N_k} [px^k(s_i^k) - w] e^{-\rho t_i^k} = 0.$$

Applying similar steps to those in the single stand case, we obtain the Lagrangian  $L = \int_0^{\infty} e^{-\delta t} U(c) dt + V + \mu B$ , which yields  $\mu = \lambda_0$  and the following equation for any cutting moment  $t_i^k$ :

$$\begin{aligned} \partial L / \partial t_i^k = & \lambda_0 e^{-\rho t_i^k} \{ pF[x^k(s_i^k)] - pF[x^k(s_{i+1}^k)] \} e^{-\rho s_{i+1}^k} - \rho [px^k(s_i^k) - w] \} \\ & + e^{-\delta t_i^k} \left\{ A[s^1(t_i^k), \dots, s^m(t_i^k)] - A[s^1(t_i^k), \dots, 0, \dots, s^m(t_i^k)] \right. \\ & \left. - \sum_{k=0}^{m-1} \int_{t_{i+1}^{k+k+1}}^{t_{i+1}^{k+k}} e^{-\delta(t-t_i^k)} \partial A / \partial s^k dt \right\} = 0, \end{aligned} \quad (28)$$

where we denote  $t_i^{m+j} = t_{i-1}^j$ . Without *in situ* valuation, i.e. with  $A \equiv 0$ , we obtain the Faustmann formula [cf. equation (14)], implying that the stands are cut independently of each other and of the forest owner's characteristics. Thus, in this case the initial age distribution of the forest is repeated in cycles, with a length equal to the rotation period common to all stands. In contrast, the *in situ* valuation breaks the separation between harvesting and consumption as well as the separation between different stands. The stands are not cut separately because harvesting any stand having *in situ* value has an impact on the forest owner's budget and thus on the cutting decision of any other stand with *in situ* value. In addition, the dependence of the *in situ* valuation of any stand on the age distribution of the whole forest forms another link between the harvesting decisions. We can also observe from (28) that any individual stand without *in situ* value, i.e. with  $\partial A / \partial s^k = 0$ , is harvested independently of other stands according to the Faustmann rotation but has an effect on the rotation ages of stands with *in situ* value through the budget constraint.

We can view equation (28) as an ordinary nonlinear difference equation. When  $\delta = \rho$ , its order depends on the number of stands and is equal to  $2m-1$ . With one stand (section 3), the rotation period in the stationary solution depends on the initial age of the stand but the rotation period is constant. Assume a case of two stands with equal initial ages. A rotation program that maintains the equality of stand ages leads to highly uneven time distribution of marginal *in situ* benefits. It is thus possible to increase the present value of *in situ* benefits by yielding a deviation in the stand ages. The costs of this deviation in the form of reduced forestry income can be made arbitrarily small if the deviation is produced slowly during several rotations. This suggests that with multiple stands there may be convergence toward some heterogeneous steady state age distribution. To take an example, assume  $\delta = \rho$  and consider a forest of two stands, in which case the equation (28) for stand 1 takes the form:

$$\lambda_0 \{ pF[x^1(s_i^1)] - pF[x^1(s_{i+1}^1)] e^{-\rho s_{i+1}^1} [px^1(s_i^1) - w] \} + A[s^1(t_i^1), s^2(t_i^1)] - A[0, s^2(t_i^1)] - \int_{t_{i+1}^2}^{t_{i+1}^1} e^{-\delta(t-t_i^1)} \partial A[s^1(t-t_i^1), s^2(t-t_{i+1}^2)] / \partial s^1 dt - \int_{t_i^1}^{t_{i+1}^2} e^{-\delta(t-t_i^1)} \partial A[s^1(t-t_i^1), s^2(t-t_{i-1}^2)] / \partial s^1 dt = 0. \quad (29)$$

Using equation (29) we can compute examples by proceeding backwards from a point arbitrarily close to a steady state rotation program consistent with a given  $\lambda_0$ . Along a steady state, both stands have equal rotation period and the period between cuttings equals half the rotation length (such a solution satisfies (29)). After computing the solution backwards, we can formulate *ad hoc* consumption preferences and initial asset level that are consistent with the given  $\lambda_0$  and the budget constraint<sup>8</sup>.

<sup>8</sup>Solving the rotation program (forward) for a given initial stand age distribution and nonforest assets requires a shooting algorithm.

An example of a rotation program converging toward an steady state is depicted in Figure 4. At  $t=0$  the age distribution is  $[0,1.6]$  years. The first rotation period for the older stand is 74 years and for the younger 91 years. After 9 rotations the harvesting periods are very near the steady state rotation period ( $\approx 83$  years). Thus in the steady state there is a harvest after each 41.5 years. The example in Figure 4 suggests a hypothesis that within a class of concave and increasing *in situ* valuation functions there occurs a convergence toward a forest with increasing heterogeneity in stand ages.

The example in Figure 5 presents another demonstration of how *in situ* valuation may change the structure of forest year classes. Without *in situ* valuation, the optimal

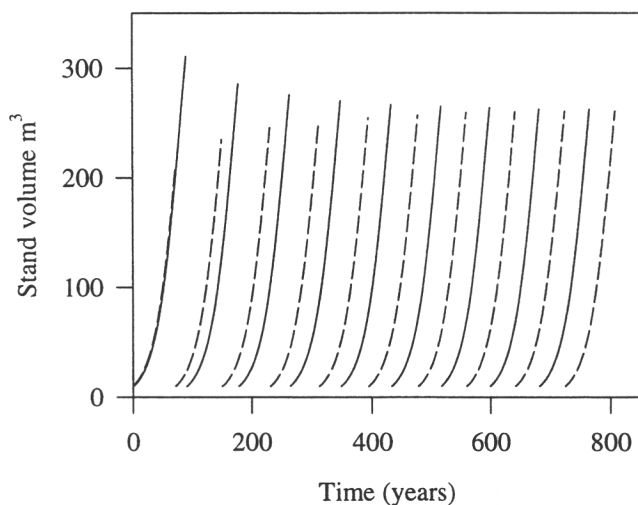


Figure 4. Optimal development of a forest with two stands.

Note:  $A(s_1, s_2) = A(a + s_1)^{-k}(a + s_2)^{-k}$ ,  $U(c) = (c^{1-\alpha} - 1)/(1-\alpha)$ ,  $x_1(s) = K/(1 - Ce^{-rs})$ ,  $C = (x_0 - K)/x_0$ ,  
 $A = 10^4$ ,  $a = 0.02$ ,  $k = 0.5$ ,  $\rho = 0.03$ ,  $\delta = 0.03$ ,  $p = 170$ ,  $w = 2000$ ,  $r = 0.048$ ,  $K = 500$ ,  $x_0 = 10$ ,  $\alpha = 1/2$ .

rotation period is 66 years and independent on whether a forest consisting two stands of the same age is viewed as a one-stand or two-stand system. We next compute the steady state rotation periods for the one-stand and the two-stand forest as functions of the *in situ* valuation parameter. For the two-stand case, it is assumed that the initial stand ages equal the steady state pattern. Increasing the *in situ* valuation parameter above 500 implies that when the forest is taken as one unit the steady state solution does not exist and it is optimal to give up forest harvesting immediately or in finite time. However, taking the two stands as individual units implies that there exists at least a locally optimal steady state solution for the two-stand system even with much higher *in situ* valuation. This suggests that the results of one-stand multiple use models are not directly applicable at the forest level. We can also hypothesize that a lower bound (e.g. legal) restriction to stand size may decrease timber supply because forest owners may leave their forests totally unharvested because harvesting may be carried out only if the stand size is smaller *ceteris paribus*.

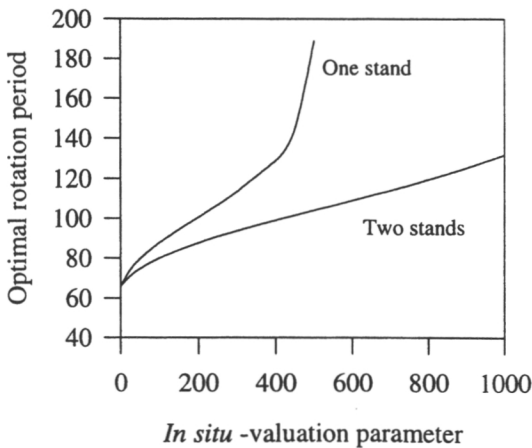


Figure 5. Harvesting a forest as one stand or as two stands.

Note:  $A(s_1, s_2) = A(a + s_1)^k (a + s_2)^k$ ,  $U(c) = (c^{1-\alpha} - 1) / (1 - \alpha)$ ,  $x(s) = K / (1 - Ce^{-rs})$ ,  $C = (x_0 - K) / x_0$ ,  $a = 0.1$ ,  $k = 1/2$ ,  $m = 1000$ ,  $a_0 = 1000$ ,  $p = 170$ ,  $r = 0.048$ ,  $\alpha = 1/2$ ; with one stand  $x_0 = 10$ ,  $K = 500$ ,  $w = 2000$ , and with two stands  $x_0 = 5$ ,  $K = 250$ ,  $w = 1000$ .

## 6. Conclusions

The literature on nonindustrial private forest owners consists of two distinct traditions: the two-period utility maximizing tradition and the rotation period, profit maximizing tradition. The two-period tradition has shown that in cases of uncertainty, imperfect capital markets or *in situ* benefits the Fisherian separation theorem does not hold, implying that forest management decisions depend on forest owner-specific factors. This dependence then fundamentally changes the properties of timber supply and e.g. results in forest taxation. However, this problem is not taken into account in the optimal rotation tradition. Our results show that a utility-based rotation model with *in situ* benefits and consumption/savings dynamics alters the most basic properties of optimal forest rotation. Among other things, the rotation period depends on the initial stand age. Even in the single stand case without any exogenous time dependent factors, the rotation period may not be constant and forest harvesting may not continue infinitely. The determination of the value of forest land differs from the results of Faustmann (1849) and becomes dependent e.g. on the property right regime related to the *in situ* benefits. Obviously the rotation model with the forest owners consumption/savings decision making should also be extended to cover e.g. price uncertainty.

## Appendix 1.

The problem (1)-(8) can be given in the following equivalent form:

$$W_1 = \max_{\{t_1, c\}} \int_0^{t_1} [U(c) + A(s)] e^{-\delta t} dt + W_2 [a(t_1^-) + px(s_1) - w, t_1] \quad (\text{A1})$$

$$\text{s.t.} \quad \dot{a} = \rho a - c + m, \quad a(0) = a_0, \quad (\text{A2})$$

$$\dot{x} = F(x), \quad x(0) = x_0, \quad (\text{A3})$$

$$\dot{s}=1, s(0)=0, \quad (\text{A4})$$

where  $W_2$  is the value function for the period  $t \in [t_1, \infty)$  problem with  $a(t_1)=a(t_1^-)+px(s_1)-w$  as the initial nonforest assets. Let  $\lambda$ ,  $\varphi$  and  $\psi$  denote the costates for nonforest assets  $a$ , the stand volume  $x$  and the age of the stand  $s$  respectively. By Seierstad and Sydsæter (1987, p. 182 theorem 3, note 2) the Hamiltonian and the necessary conditions for the optimal solution are:

$$H=U(c)+A(s)+\lambda(\rho a-c+m)+\varphi F(x)+\psi, \quad (\text{A5})$$

$$U'(c)-\lambda=0, \quad (\text{A6})$$

$$\dot{\lambda}=\lambda(\delta-\rho), \quad (\text{A7})$$

$$\dot{\varphi}=\varphi[\delta-F'(x)], \quad (\text{A8})$$

$$\dot{\psi}=-A'(s)+\psi\delta, \quad (\text{A9})$$

$$\lambda(t_1^-)=e^{\delta t_1} \partial W_2 / \partial a(t_1^-), \quad (\text{A10})$$

$$\varphi(t_1)=e^{\delta t_1} \partial W_2 / \partial x(s_1), \quad (\text{A11})$$

$$\psi(t_1^-)=0, \quad (\text{A12})$$

$$H_1(t_1^-)=-e^{\delta t_1} \partial W_2 / \partial t_1. \quad (\text{A13})$$

To develop conditions (A10), (A11) and (A13) we can apply theorem 9 in Seierstad and Sydsæter (1987, p. 213). We obtain  $e^{\delta t_1} \partial W_2 / \partial a(t_1^-)=\lambda(t_1)$  and, by equation (A10), that  $\lambda(t_1^-)=\lambda(t_1)$ , i.e.  $\lambda$  (and  $c$ ) must be continuous. Because  $a(t_1^-)+px(s_1)-w$  is the level of nonforest assets at  $t_1$ ,  $e^{\delta t_1} \partial W_2 / \partial x(s_1)=p\lambda(t_1)$  and equation (A11) yields  $\varphi(t_1^-)=p\lambda(t_1)$ . Equation (A13) yields  $H(t_1^-)=H(t_1)$ , which implies, after canceling the terms that remain continuous at  $t_1$ , that  $A(s_1)+p\lambda(t_1)F[x(t_1^-)]=A(0)+\lambda(t_1)\rho[px(t_1^-)-w]+\varphi(t_1)F(x_0)+\psi(t_1)$ . To proceed we must develop the term  $\varphi(t_1)F(x_0)$ . Equations  $dx/dt=F(x)$  and  $d\varphi/dt=\varphi[\delta-F'(x)]$  imply that  $d\varphi/dx=\varphi[\delta-F'(x)]/F(x)$ , which is a differential equation in  $x$ :

$$\varphi(x) = \varphi(x_0)e^{\int_{x_0}^x [\delta-F'(x)]/F(x) dx} = \varphi(x_0)e^{\int_{x_0}^x \delta/F(x) dx} e^{-\int_{x_0}^x F'(x)/F(x) dx}$$

$$= \varphi(x_0) e^{\int_{t_1}^t \delta dt} e^{\int_{x_0}^x \ln[F(x)]} = \varphi(x_0) e^{\delta(t-t_1)} F(x_0)/F(x).$$

Thus  $\varphi(t) = \varphi(t_1) e^{\delta(t-t_1)} F(x_0)/F(x(t))$ , implying  $\varphi(t_1) F(x_0) = \varphi(t_2) e^{-\delta s_2} F[x(s_2)]$ . More generally, we obtain, for any  $i=1, \dots, \infty$ , that  $\varphi(t_i) F(x_0) = p \lambda_0 e^{(\delta-\rho)t_{i+1}} F[x(s_{i+1})] e^{-\delta(t_{i+1}-t_i)} = p \lambda_0 e^{-\rho t_{i+1} + \delta t_i} F[x(s_{i+1})]$ . Next by  $\psi(t_i) = \int_0^{s_{i+1}} A'(s) e^{-\delta s} ds$  equation  $H(t_i^-) = H(t_i)$  simplifies to:

$$\lambda_0 e^{(\delta-\rho)t_i} \left\{ pF[x(s_i)] - \rho [px(s_i) - w] - pF[x(s_{i+1})] e^{-\rho s_{i+1}} \right\} + A(s_i) - A(0) - \int_0^{s_{i+1}} A'(s) e^{-\delta s} ds = 0,$$

which is identical to equation (13).

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