METSÄNTUTKIMUSLAITOKSEN TIEDONANTOJA 785, 2000
FINNISH FOREST RESEARCH INSTITUTE, RESEARCH PAPERS 785, 2000

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16.11.00

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Penttinen, Markku. 1999. Timber harvesting with variable prices, costs and interest rates. Metsäntutkimuslaitoksen tiedonantoja 785. Finnish Forest Research Institute, Research Papers 785. ISBN 951-40-1751-X, ISSN 0358-4283. 38 p.

Publisher: Finnish Forest Research Institute

Approved: Kari Mielikäinen, Research Director, 31.8.2000.

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Distribution: Library of Finnish Forest Research Institute, P.O. Box 18, FIN-01301 Vantaa, Finland. Tel. +358 9 8570 5582, Email: library@metla.fi.

Acknowledgement

The author is greatly indebted to Prof. Jari Kuuluvainen, Prof. Markku Ollikainen, Prof. Olli Tahvonen, Dr. Pekka Ripatti, PhD. Student Kari Hyytiäinen and two anonymous reviewers for their valuable comments on the manuscript.

Abstract

This papers solves the optimal harvesting time problem of a non-industrial private forest (NIPF) owner who typically has a forest management plan and merchantable forest stands. The optimal harvesting time is defined in a volatile market situation.The infinite period problem is also formulated to allow for variable stumpage prices and reforestation costs in a two-period framework, the first of which covers the near future with *dynamic* price and cost functions and the second the rest of the infinite future with *trend* price and cost functions.

The existence and uniqueness of an optimal policy is demonstrated on the basis of the explicit quasi-concavity of the objective functions. First, the solutions are constructed with prices and costs dependent on stand age only. Both cases in which the same prices and costs hold for all periods and cases in which there are dynamic prices and costs in the first period and trend ones in subsequent periods are considered. Second, the age-dependent functions are multiplied separately by the calendar time dependent exponential terms. Solutions are provided both in the case with the same age-dependent functions and the case with dynamic functions for the first period and trend functions for the subsequent periods

The sensitivity and comparative static analyses are studied with respect to the interest rate, price and cost changes, both analytically and numerically. Optimal rotation solutions are presented with alternative competing volume growth functions. Final results are provided by a gross income growth function. Competing optimisation models are discussed, and alternative volume growth models and a value growth model are compared.

The key notion of the research is the sensitivity and comparative static analysis of the optimal rotation solutions with respect to roundwood prices, reforestation costs and interest rates. Different local market parameter and alternative growth data estimates are applied in testing the impact of price, cost and interest rate parameters. The purpose of the study is to provide tools for day-to-day decision-making in the changing world of forestry and also to compare silvicultural recommendations with the solutions.

Many NIPF owners have a tendency to try to sell only during peak price periods. Their behaviour is compared with policy results obtained using empirical data on the turbulent market place with fluctuating prices, growth models and the optimal rotation models developed.

Key words: optimal rotation, variable prices, costs and interest rates, sensitivity analysis, comparative statics, existence and uniqueness of global optimum solutions.

Contents

1 Introduction

1.1 Competing views in forest economics

The history of optimal rotation research includes solutions based on different approaches. A traditional misunderstanding was to define the optimal rotation age of the standing timber as in the case of wine ageing, and thereby to forget the necessity of felling the forest before the land can be used again. Other such examples include biologically maximum sustained yield (MSY), which ignores economic evaluation, and sustained yield forestry (Waldreinertragswirtschaft), which assumes that the interest rate is zero (Löfgren 1990). In addition, a widely applied criterion, the rate of growth of capital, known by terms such as the internal rate of return (IRR), has been shown to be incorrect by Samuelson (1976). [1]

In forestry, the marginal rate of return approach has been supported by Duerr, among others (see e.g. Duerr et al. 1979). Even the local tradition applied to forestry practice was, according to Duerr, to use a predefined marginal interest rate as suggested by e.g. Nyyssönen (1958) and Nyyssönen (1997). This notion is inspired by the portfolio management approach and the theory of interest [2]. For example, when cutting the forest the "average" rate of growth is replaced by a *marginal* rate of growth (Fisher 1954, p. 165). Moreover, timber and timber investments are capital goods, and they should be managed at a rate of return equal to the return on other capital investments in the economy, i.e. at the market interest rate according to Hirshleifer (1974). Some proposals, as, for example, in the case of "financial rotation periods", are, however, based on maximisation of *average* relative profitability, in which the value of both stocks and land form bounded capital (Speidel 1984, p. 172).

In forest economics, however, the main approach has been to use the net present value (NPV) as the objective variable of the optimisation process. The exclusive position of the König-Faustmann NPV has been criticised by Oderwald & Duerr (1990), among others. Their criticism, in turn, has been countered by Chang (1990) and Klemperer (1996). On the whole, König-Faustmann NPV maximisation based on the arguments presented in the classic article by Samuelson (1976) still remains the cornerstone of the standard approach (see also Newman 1988). [3]

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[1] Recall that, when using IRR maximisation, one hypothesis is that the area of land available for forestry is infinite and that access to all capital markets is closed (Newman 1988). The IRR criterion may actually lead to unsatisfactory conclusions as regards either the infinite or zero wealth increment (Hirshleifer 1970).

[2] In firms, the primary criterion of decision making is profitability. It is based on the theory of interest (Fisher 1931) and the return on investment (ROI) that follows from it. Profitability is considered to be the best available measure of efficiency (see e.g. Brozik 1984). The traditional interest theory approach, as outlined by Fisher (1931, 1954, p. 159), requires that the rate of return over cost must exceed the rate of interest.

[3] Note that the PV maximisation solution of rotation is well-defined provided that (i) the capital market is perfect, (ii) the future price of timber is known, (iii) forest land can be bought and sold in a perfect market and (iv) future technical lumber yields are known (Löfgren 1990).

1.2 Previous work on deterministic optimal rotation modelling

This section considers the theoretical foundations of timber harvesting. Faustmannian deterministic[4] net present value (NPV) results are provided by numerous books, e.g. Johansson & Löfgren (1985). The "Faustmann" (NPV) solution, the "Fisher" (ROI) solution, internal rate of return (IRR) and maximum sustained yield (MSY) have been compared by Samuelson (1976). Rideout (1986) compared the Fisher and Faustmann solutions and also suggested benefit-cost ratio maximisation. The differences between the optimal rotation lengths obtained when applying the Faustmann formula and maximum-sustained yield have been studied by Binkley (1987), among others. Löfgren (1990) summarised the 'profitability war', focusing on PV and land rent approaches, and summarised conditions under which the optimal rotation problem is well-defined. However, the König-Faustmann tradition has been criticised by Oderwald & Duerr (1990), who suggest optimising the firm's investment in timber stock in terms of marginal revenues and cost per unit of capital, an approach that has provoked more criticism than support. The state of the art in optimal rotation has been summarised by Newman (1988).

McConnel et al. (1983) presented optimal rotation solutions for subsequent rotations with exponential prices and costs[5]. Hardie et al. (1984) used the dynamic programming approach and solved an optimal rotation problem with price, cost and yield forecasts assuming steady state rotations after some fixed number of rotations. Newman et al. (1985) analysed optimal rotation solutions by linking subsequent rotations in the case of evolving prices and solved the problem with exponential prices. Yin & Newman (1995) considered a case where prices and costs increase exponentially, and either of which can grow faster than the other. Chang (1998) defined optimal solutions relating to subsequent rotations and provided numerical solutions with variable prices.

Comparative static analysis is one of the main approaches applied in optimal rotation research. Chang (1982) studied the impact of different forest taxation systems on optimal rotation age. He has also dealt with the influence of different factors such as price, the interest rate, regeneration cost and taxation on the rotation age (Chang 1983). Chang (1984) developed the sensitivity results in his comparative static analysis. The response of optimal rotation and management intensity to changes in price, management cost and the discount rate has been analysed by Nautiyal & Williams (1990).

The maximum principle and the control theory approach has been applied to the Faustmann formulation. A steady state solution for forestry management problems with e.g. variable harvesting costs has been advocated by Heaps (1984). Recently, *in situ* versions of optimal forest rotation models have been analysed using the control

[4] Stochastic optimal rotation models have been actively researched, many contributions applying stochastic differential equations (see e.g. Brazee & Mendelson 1988, Yin & Newman 1997). A stochastic NPV solution for a harvest strategy has been proposed by Gong (1999). However, stochastic optimal rotation results are not discussed here as they will be presented in a complementary publication.

[5] The cost of the regeneration delay has been assessed by Brodie & Tedder (1982) and Lappi (1983), among others.

theory approach by Kuuluvainen & Tahvonen (1999) and Tahvonen & Salo (1999).

Growth and yield functions and the forest management planning (FMP) tradition form the cornerstones for the implementation of results. The optimal rotation problem has been scrutinised for practical purposes by Nyyssönen (1958) and Nyyssönen (1997) on the basis of marginal rate of return. The rotation dilemma has been studied as part of a forest stand planning problem by Kilkki (1968) and as a dynamic programming problem by Kilkki & Väisänen (1969). Local growth functions have been investigated by Nyyssönen & Mielikäinen (1978), as well as by Vuokila & Väliaho (1980), among others. Fridh & Nilsson (1980) have suggested a simplified growth function. The foundations of optimal rotation have been presented by Vuokila (1980), among others. The relationship between the value increment and the volume increment has been analysed by Nyyssönen & Ojansuu (1982).

Recommendations concerning the optimal rotation for forestry practice have been published by Forestry Centre Tapio (1994). The Finnish Ministry of Agriculture and Forestry (1997) has issued a decision on the prerequisites for regeneration cutting as part of implementation of the Forest Act. Criteria for clearcutting based on legal provisions and recommendations have been compared with optimal rotation results by Hyytiäinen & Tahvonen (2000).

1.3 The aim of this study

A non-industrial private forest (NIPF) owner typically has a valid forest management plan (FMP) and merchantable stands, some of which he/she wants to harvest.

(i) First, this study seeks to support the timber harvesting decisions of the forest owner. The FMP could be extended to include, say, the order in which the merchantable forest stands should be harvested and even harvesting time recommendations as functions of changes in interest rates, prices and costs etc.

(ii) Second, the empirical part of the study applies different volume and income yield functions and local data in order to review the restrictions of the Forest Act (Ministry of the Agriculture and Forestry 1997) and the forest management recommendations of Forestry Centre Tapio (1994).

(iii) Third, the study aims to produce results for implementing decision support system (DSS) personal computer (PC) programs for day-to-day decision-making. Using FMP data as input, the forest owner could then even study dynamic situations with varying interest, price, cost etc. parameters. Sensitivity and comparative static analyses are used to describe the interactions between the various features of the planning situation.

(iv) Last but not least, the rationale of forest owners' roundwood sales behaviour is assessed. Although they primarily follow the stumpage price, forest owners also pay close attention to developments in silvicultural costs and interest rates. There is a discussion of the practice whereby many forest owners try to sell only when prices are at their peak.

This study is based on the dynamic rotation modelling contributions of McConnell et al. (1983), Hardie et al. (1984), Newman et al. (1985), Yin & Newman (1995), and Chang (1998). Previous results have been extended by allowing parameters and functions to vary and using quasi-concavity instead of concavity. Both optimal solutions and sensitivity analysis are based on local economic and yield data.

2 Data and Methods

2.1 The optimal rotation problem definition

The traditional Faustmann optimal rotation, also called the harvest time problem, can be formulated in a continuous time framework as the moment when the value of continued growth equals the opportunity cost of waiting (Brazee & Mendelson 1988):

$$p\,q'(\tau) = r\,p\,q(\tau) + r\;b, \tag{1}$$

where p is the constant stumpage price, b is the value of the bare land, r is the interest rate and $q(\tau)$ is the growth and yield function describing the timber volume at *age* τ [6]

In the above formulation (1), with the future condensed into one concept, the value b of the bare land poses a two-period problem. The basic discrete deterministic problem of optimal rotation with variable prices $p(t)$ and costs $c(t)$ is that of dynamic programming (DP) as applied by Hardie et al. (1984) (Amidon & Akin 1968 have used DP earlier for choosing optimal thinning rules; see also Gong 1992 and Filius & Dul 1992, Salminen 1993):

$$w_n(T_n) = \max_t \{ [p(t)q(t) - c(t)]\exp(-rt) + w_{n+1}(t,T_{n+1})\exp(-rt) \} - c(0). \tag{3}$$

The regeneration cost $c(0)$ of the present generation cannot be affected and is ignored in (3). Note that the time t and the age τ_n of the tree generation n to be harvested are the same, $t = \tau_n$. The price and cost functions may change over calendar time t, but the growth is nevertheless a function of biological age $\tau_n, \tau_{n+1}, \tau_{n+2}, \ldots$ The optimal harvesting age is $T_n, T_{n+1}, T_{n+2}, \ldots$ The growth and yield function q actually depends both on the age τ and planting density m (see Chang 1983), but for the present generation some density has already been chosen, i.e. $q(\tau,m) = q(\tau,.)$ and is denoted by $q(\tau)$. It depends on age and on a number of forestry variables such as the stand basal area, the basal area median diameter and the dominant height (local yield models are presented in Pukkala & Miina 1997).[7]

The DP requires *separability* and *monotonicity* (see Nemhauser 1966), the latter meaning that $w_{n+1}(t,T_{n+1})\exp(-rt)$, which depends on t, is monotonic.

With available information as the crucial limiting factor, it is clear that information concerning stumpage prices $p(t)$, silvicultural costs $c(t)$ and interest rates r in the

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[6] The steady state problem can be defined as wealth maximisation according to Binkley (1987)

$$\max_\tau w(\tau) = -c + p\,q(\tau)\exp(-r\,\tau) + w(\tau)\exp(-r\tau), \tag{2}$$

where $w(\tau)$ is wealth as a function of rotation age τ and c is the constant regeneration cost. The traditional soil expectation value (SEV), calculated as the sum of all discounted future revenues, has been presented in forest economics books such as Johansson & Löfgren (1985).

[7] Recall that, according to empirical studies, growth is quite deterministic compared with price (Lausti & Penttinen 1998).

near future is far better known than that for subsequent rotations. Actually, the information available after the present rotation period is very limited as regards fluctuations. Thus considering only non-increasing functions $w_{n+1}(t,T_{n+1})$ for the subsequent period does not limit the applicability of the model. This means that the monotonicity of $w_{n+1}(t,T_{n+1})\exp(-rt)$ is fulfilled.

2.2 Optimal rotation solutions

A key question for the general deterministic case is the discrepancy between calendar time t and age τ, i.e. whether the future NPV in (3), $w_{n+1}(t,T_{n+1})$, depends on t or not. If harvesting the present generation n is delayed, the beginning of the next generation is also delayed in calendar time t. Thus the deterministic model with variable parameters consists of two parts:

(a) All prices, profit ratios and costs depend only on age τ $[w_{n+1}(t,T_{n+1}) \equiv w_{n+1}(T_{n+1})]$:

First, consider an extension of the traditional approach that allows the price $p(\tau)$ of timber to vary according to the biological age of the stand (Nautiyal & Williams 1990 and many others assume the price p to be constant). Sales revenue $p(\tau)q(\tau)$ may be subject to taxes and other deductions so that the forest owner keeps only a proportion $d(\tau)$ of sales revenue. These charges, the share $1-d(\tau)$, include items such as capital income tax, which is currently 29% in Finland (for the impact of taxation on optimal rotation, see Chang 1982), marketing and logging costs etc. Whenever the regeneration costs $c(0)$ of the present generation are also included, the soil expectation value (SEV) is denoted by $V(\tau)$ and is

$$V(\tau) = [\,p(\tau)q(\tau)d(\tau)\exp(-r\,\tau) - c(0)] \,/\, [1 - \exp(-r\tau)], \tag{4}$$

where varying prices $p(\tau)$ and the net profit ratio $d(\tau)$ of the present generation, the growth and yield function $q(\tau)$ and constant reforestation costs[8] $c(0)$ have been used. The optimal rotation period T is then defined (see Appendix) by

$$p'(T)/p(T) + q'(T)/q(T) + d'(T)/d(T) = r\,[1 - c_r(T)] \,/\, [1 - \exp(-rT)], \tag{5}$$

where notation $c_r(T)$ denotes the relative reforestation cost, i.e. the ratio $c_r(T) = c(0)/[p(T)q(T)d(T)]$.

[8] This approach to using $c(0)$ suffers from the necessity of having to estimate parameters which have a time difference of e.g. 80 years time, as in the case of $c(0)$ and $p(T)$. Moreover, $c(0)$ is not relevant for present period costs.

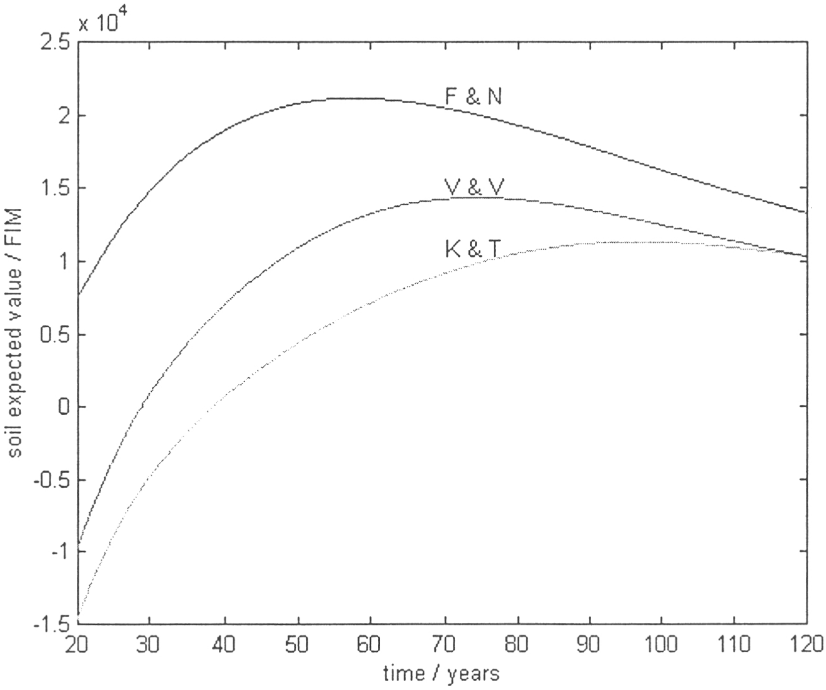

Figure 1. *The soil expectation value S FIM/ha, when r = 3%, p(T) = FIM 240/m³, p'/p = 1%, d(T) = 0.71, d' = 0, c(T) = FIM5000/ha, c'/c = 0.5% and growth and yields are those in (16) of Fridh & Nilsson, (17) of Kuuluvainen & Tahvonen and (18) of Vuokila & Väliaho.*

The maximum can be shown to be the global maximum that is already based on the explicit quasi-concavity of $V(\tau)$ for some τ_0, $\tau_0 > 0$ (see Appendix). Loosely speaking, quasi-concavity means that a function is first increasing then decreasing. Unfortunately, $V(\tau)$ is not even quasi-concave for the whole positive axis. Moreover, c(0) is not necessarily a correct estimate for $c(T_n)$, $c(T_{n+1})$,[9] … . Consequently, the alternative of using c(0) as the regeneration cost is rejected here, and hence $V(\tau)$ is subsequently rejected as a relevant model for the analysis.

Second, varying reforestation costs can be included in the optimal rotation model using the explicit reforestation costs c(t) of the next generation.[10] The soil expectation value (SEV) denoted by $S(\tau)$ is then

$$S(\tau) = \{[p(\tau)q(\tau)d(\tau) - c(\tau)] \exp(-r\tau)\} / \{1 - \exp(-r\tau)\} - c(0) . \tag{6}$$

The corresponding optimal rotation period T is defined by

[9] Note that local law already requires the reforestation of the next rotation to be performed within 5 years of the final felling.

[10] In practice, regeneration takes place 2–5 years after harvesting, which is recognised in c(t), but not shown explicitly in the function form.

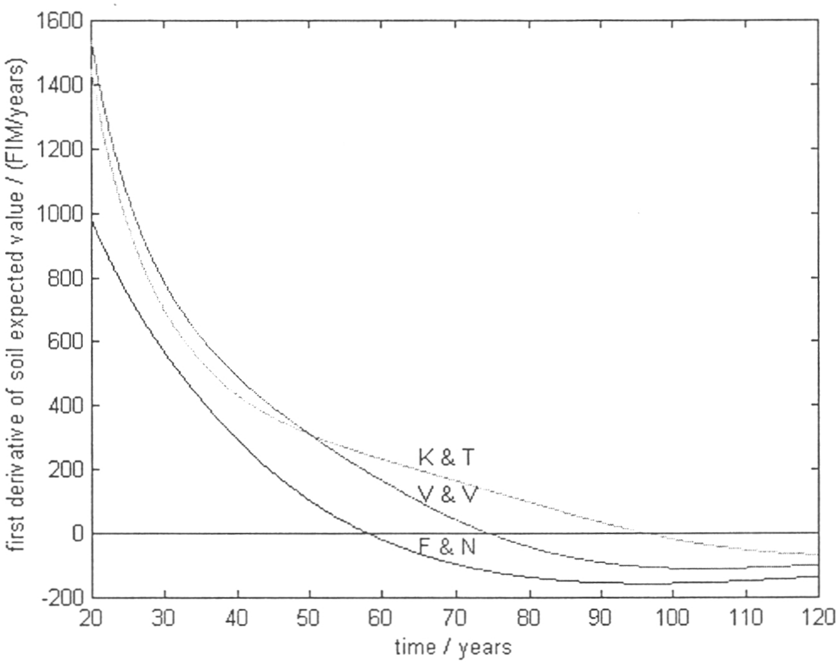

Figure 2. *The derivative S' FIM/(ha year) of the soil expectation value ,when r = 3%, $p(T)$ = FIM 240/m^3, p'/p = 1%, $d(T)$ = 0.71, d' = 0, $c(T)$ = FIM5000/ha, c'/c = 0.5% and the growth and yield are those in Kuuluvainen & Tahvonen, Vuokila & Väliaho and Fridh & Nilsson.*

$$p'(T)/p(T) + q'(T)/q(T) + d'(T)/d(T) = c_r(T)\, c'(T) / c(T)$$

$$+ \; r\,[1 - c_r(T)] / [1 - \exp(-rT)], \tag{7}$$

where $c_r(T)$ denotes the reforestation cost ratio $c_r(T) = c(T)/[p(T)q(T)d(T)]$. The extremum is now also shown to be the global maximum based on the explicit quasi-concavity of $S(\tau)$, which allows at most a single sign change pattern, from + to –, of the derivative $S'(\tau)$ (see Appendix).

(b) The prices, costs and profit ratios depend on calendar time t $[w_{n+1}(t,T_{n+1})]$:

First, a two-period approach is considered. For periods n+1, n+2, … the SEV is that of (6) $S(T_{n+1})$ in which $c(0)$ is ignored. Recall that Hardie et al. (1984) proposed a solution in which only steady state rotations appear after k rotations. Note that the optimal rotation age $T_{n+1}, T_{n+2}, \ldots$ is the single solution obtained from (7), $T_{n+1} = T_{n+2}$, … , because the trend price $p(\tau)$, net profit ratio $d(\tau)$ and cost functions $c(\tau)$ of periods n+1, n+2, … are the same for all subsequent periods. Now the SEV denoted by $W(t)$ including all periods n, n+1, n+2, … is

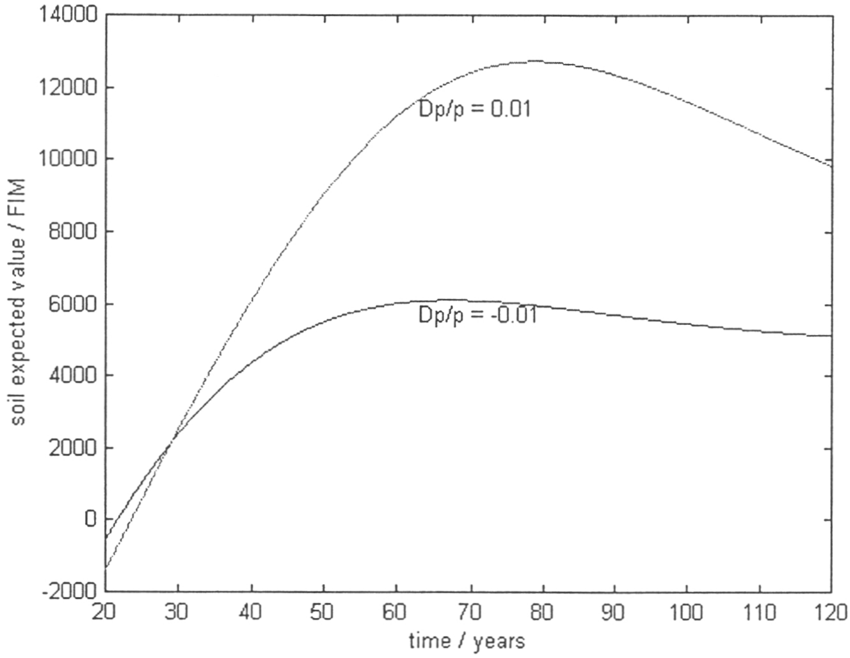

Figure 3. *The soil expectation value W FIM/ha, when r = 3%, π (T) = FIM 240/m^3, δ(T) = 0.71, δ'/δ = −0.05%, γ(T) = FIM5000/ha, γ'/γ = 0.5%, c'=0, d'=0 and the yield is that in (18) of Vuokila & Väliaho with price changes π'/π = −1% and +1%, p'/p = 0.4%.*

$$W(t) = c(0) + [\pi(t)q(t)\delta(t) - \gamma(t)]\exp(-rt) + S(T_{n+1})\exp(-rt), \quad (8)$$

where the "dynamic" price $\pi(t)$, the net profit ratio $\delta(t)$ and the reforestation cost $\gamma(t)$ functions are associated only with the near future and with the first period n. Note that the separability, in fact additivity, and monotonicity, in this case non-increasingness, required by dynamic programming (DP) hold.

The optimal rotation age $T = T_n$ is defined by

$$\pi'(T)/\pi(T) + q'(T)/q(T) + \delta'(T)/\delta(T) - [\gamma'(T)/\gamma(T)]\,\gamma_r(T) =$$

$$r\,\{\,1 - \gamma_r(T) - S(T_{n+1}) \,/\, [\pi(T)q(T)\delta(T)]\,\}, \quad (9)$$

where $\gamma_r(t)$ denotes the relative reforestation cost, $\gamma_r(t) = \gamma(t)/[\pi(t)q(t)\,\delta(t)]$.

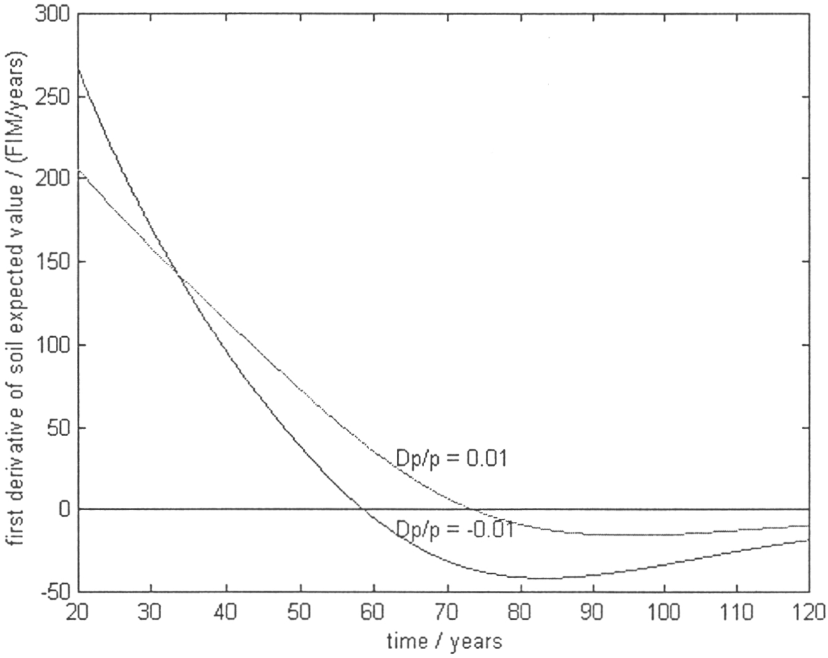

Figure 4. *The derivative of the soil expectation value W FIM/(ha year) , when r = 3%, π (T) = FIM240/m^3, δ(T) = 0.71,* δ'/δ = −0.05%, γ (T) = FIM5000/ha, γ'/γ = 0.5%, , c' = 0, d'=0 and the yield is that in (18) of Vuokila & Väliaho with price changes π'/π = −1% and +1%, p'/p = 0.4%.

The problem solution has two phases, the first of which consists of the definition of T_{n+1} as a solution of (7) in terms of the trend price p(τ), net profit ratio d(τ), cost c(τ) and yield q(τ) functions, after which the calculation of S(T_{n+1}) is as defined in (6), but without c(0).

In the second phase S(T_{n+1}) is considered constant and the optimal rotation period T = T_n is calculated using (9). The optimum exists and is unique, as shown in the Appendix.

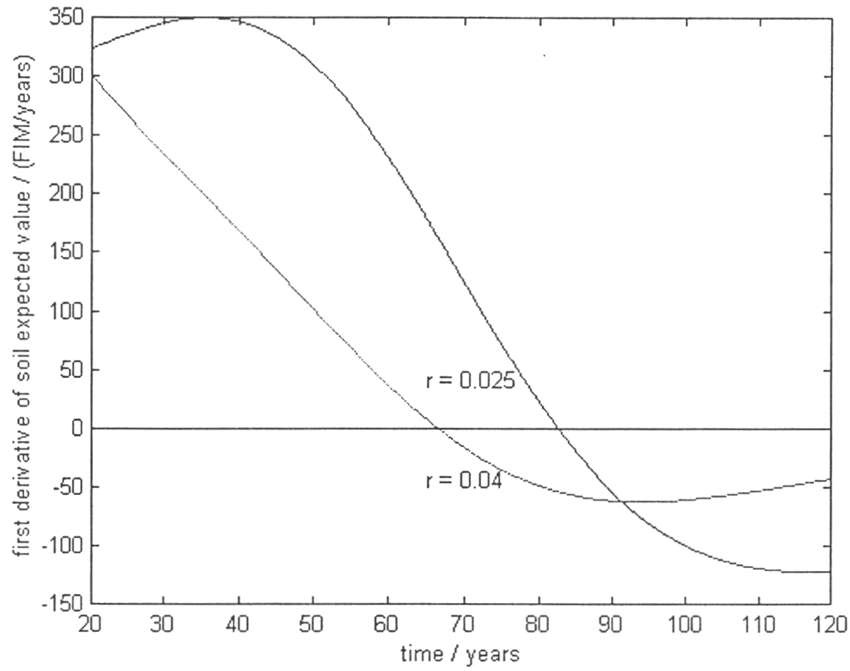

Figure 5. *The derivative of the soil expectation value W , when $\pi(T)$ = FIM 240/m^3, $\pi'/\pi = p'/p =$ 1%, $\delta(T) = 0.71$, $\delta'/\delta = -0.05\%$, $d'=0$, $\gamma(T)$ = FIM5000/ha, $\gamma'/\gamma = 0.5\%$, $c'=0$ and the yield is that in (18) of Vuokila & Väliaho with interest rates $r =$ 2.5% and 4%.*

Second, let all the price, net profit ratio and cost functions of all periods vary in calendar time t. Limitations in empirical estimates for future prices and costs suggest that, for periods n+1, n+2, ..., simple functions such as exponential ones are sufficient for the applications. Recall also that exponential price and cost functions have already been applied by McConnel et al. (1983), and developed by Yin and Newman (1995).

Inspired by both the limitations in the availability of the estimates and previous studies, the model in which calendar time emerges only in the exponential part of the functions, i.e. $\pi(t) = p(\tau)\exp(g_p\ t)$, $\delta(t) = d(\tau)\exp(g_d\ t)$, $\gamma(t) = c(\tau)\exp(g_c\ t)$, is considered. If the growth rates g_p, g_d, g_c satisfy the relation $g_p + g_d = g_c$, the solution is trivial and obtained by replacing r by $r - (g_p + g_d)$. Denote for a while $r - g_p - g_d$ by α, $r - g_c$ by β and the net income or benefit $p(\tau)q(\tau)d(\tau)$ by $b(\tau)$.

In the first phase of the calendar variable functions model the functions of period n are of the same form as those of periods n+1, n+2, ... All the optimal rotation periods are the same as shown by applying dynamic programming (DP) and the induction axiom (see Appendix), i.e. $T = T_n = T_{n+1} = \ldots$ and the soil expectation value (SEV) is then of a kind of simplified form

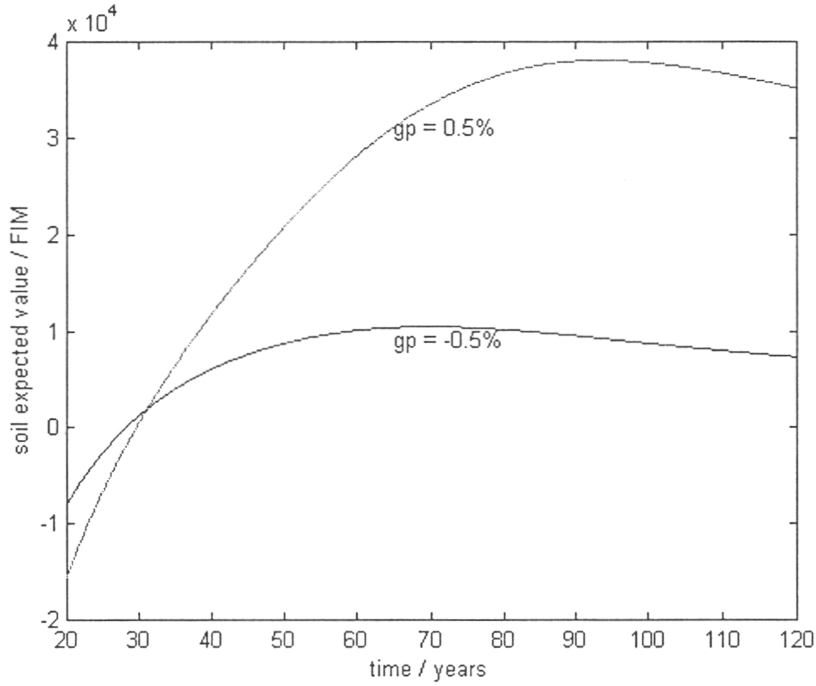

Figure 6. *The soil expectation value Z_s FIM/ha, when $r = 3\%$, $p(T)$ = FIM 240/m^3, $p'/p = 1\%$, $d(T) = 0.71$, $d' = 0$, $c(T)$ = FIM5000/ha, $c'/c = 0.5\%$ and the yield is that in (18) of Vuokila & Väliaho.*

$$Z_s(t) = c(0) + b(t)/[\exp(\alpha t)-1] - c(t)/[\exp(\beta t)-1] . \tag{10}$$

Recall that Hardie et al. (1984) produced variable rotation lengths. Whenever the periods are equal the solution of $T = T_n = \ldots$ is defined by

$$\{ b'(T) - \alpha\, b(T)/[1-\exp(-\alpha T)] \} / \{\exp(\alpha T)-1\} =$$

$$\{ c'(T) - \beta\, c(T)/[1-\exp(-\beta T)] \} / \{\exp(\beta T)-1\} \tag{11}$$

which can be solved using numerical methods. Note that the convergence requires that both $\alpha = r - g_p - g_d$ and $\beta = r - g_c$ are positive. The existence and uniqueness of the solution are discussed in the Appendix.

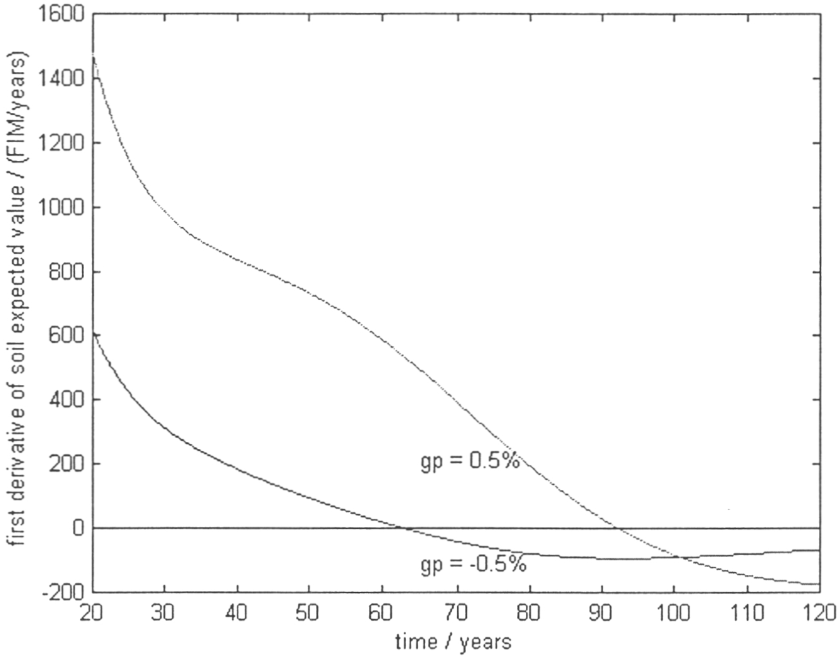

Figure 7. *The derivative Z'_s FIM/ha year of the soil expectation value, when r = 3%, p(T) = FIM 240/m^3, p'/p = +0.5 and –0.5%, d(T) = 0.71, d' = 0, C(T) = FIM5000/ha, c'/c = 0.5% and the yield is that in (18) of Vuokila & Väliaho.*

The above solution can be also be used as that for T_{n+1}, T_{n+2} = ... and the first period might have "dynamic" functions $\pi(t)$, $\delta(t)$ and $\gamma(t)$ different from those of subsequent periods. The solutions are provided[11] although they are not applied in calculations.

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[11] Denote $\pi(t)q(t)\delta(t)$ by $\varphi(t)$. The first period optimal rotation $T = T_n$ would then be defined by

$$[\varphi'(T) - \alpha\,\varphi(T)]\exp(-\alpha T) - [\gamma'(T) - \beta\,\gamma(T)]\exp(-\beta T) =$$

$$\alpha\exp(-\alpha T)\,B(T_{n+1}) - \beta\exp(-\beta T)\,C(T_{n+1}), \tag{12}$$

where B and C represent the income and cost components of $Z_s(T_{n+1})$ for n+1, n+2, ... , in (10), and are defined by $B(T_{n+1}) = b(T_{n+1})/[\exp(\alpha T_{n+1})-1]$ and $C(T_{n+1}) = c(T_{n+1})/[\exp(\beta T_{n+1}) - 1]$, whenever equal length optimal rotation periods, $T_{n+1} = T_{n+2}$, ... , are applied. Recall that in the calculation of T_n, c(0) of $Z_s(T_{n+1})$ in (10) is now ignored. Conditions under which the solution exists and is unique are discussed in the Appendix.

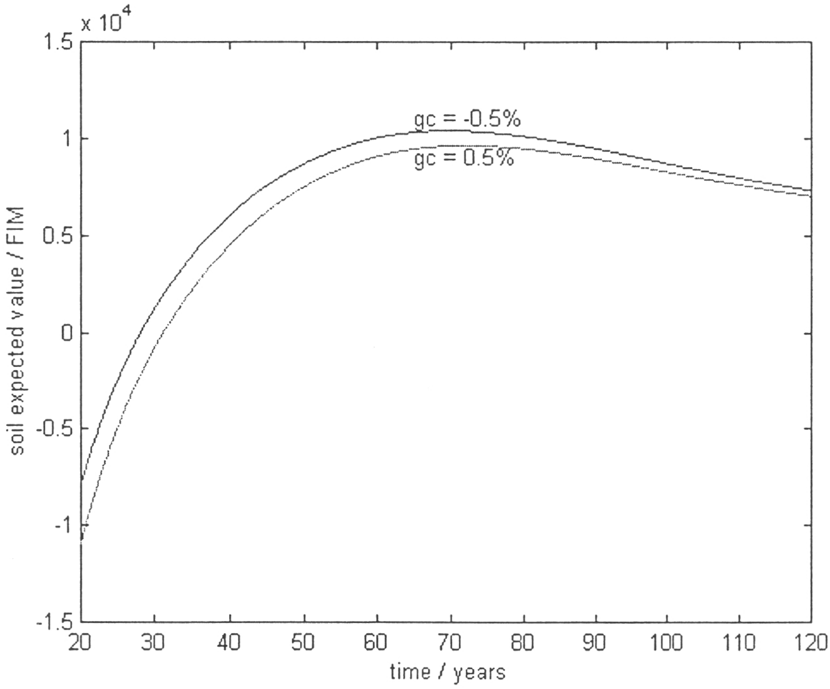

Figure 8. *The soil expectation value Z_S, when $r = 3\%$, $p(T)$ = FIM 240/m^3, p'/p = 1%, $d(T) = 0.71$, $d' = 0$, $c(T)$ = FIM5000/ha, c'/c = +0.5% and –0.5% and the yield is that in (18) of Vuokila & Väliaho.*

As expected, the cost change is of minor importance. By contrast, the interest rate has a major effect so that the 2.5% level exceeds the prerequisites of the Ministry of Agriculture and Forestry (1997) and is close to the recommendations of Forestry Centre Tapio (1994), whereas e.g. the 4% level suggests rotations that are clearly illegal.

Surprisingly, the impact of a price change is even more dramatic than that of an interest rate change, which suggests that forest owners should postpone roundwood sales as long as prices are increasing, i.e. sell when prices are at their peak.

2.3 Comparative static and sensitivity analyses

Comparative statics and sensitivity are key issues in analysing optimal policies. Risk and return studies demonstrate that the volatility of the return arises from the price component (Lausti & Penttinen 1998). The forest owner is a price taker, but he/she can speculate with regard to the optimal roundwood selling time.

The interest rate r may change. As a matter of fact, it has traditionally been the focus of sensitivity and comparative static studies.

An analysis of the proposed solutions with respect to the economic parameters, especially prices and interest rates but also costs, is relevant for forest owners. In this study, they are investigated both *analytically at the optimum rotation time* T and *numerically in the area of the optimum*. The SEVs $S(t)$ in (6) are used to establish analytically the sensitivity measures at the optimal rotation time T by differentiating the implicit function $S'(T) = 0$. Recall the partial derivative $\partial T/\partial r = -(\partial S'(T)/\partial r)/(\partial S'(T)/\partial T)$, etc.

First, the partial derivative of the optimum T with respect to the interest rate r, $\partial T/\partial r$, is considered. Derivative $\partial T/\partial r$ is (see Appendix):

$$\partial T/\partial r = f(T)\{1 - rT/[\exp(rT)-1]\} / \{f''(T)[1-\exp(-rT)] - r^2 f(T)\}, \qquad (13)$$

where $f(T) = p(T)q(T)d(T)-c(T)$. Note that the numerator is positive and the denominator negative, and thus the comparative statics $\partial T/\partial r < 0$ results .

Recall the case shown in Figure 1 with the yield function (18) of Vuokila & Väliaho (1980), which means that the value growth has not yet been recognised. In the area of the optimum rotation time, the rotation decreases by 0.8 years for every 0.1%–point increase and roughly 8 years for percentage point increase in the interest rate r above 3% (Table 1). Numerical sensitivity studies are performed using both $S(t)$ in (6) and $W(t)$ in (8).

Table 1. *Sensitivity of S- and W-function with respect to the interest rate r.*

Interest rate r / %	Optimal rotation time of S-function / years	Optimal rotation time of W-function / years
2.0	78.7	82.3
2.1	78.0	81.4
2.9	71.9	74.5
3.0	71.1	73.7
3.1	70.3	72.9
3.9	64.1	66.3
4.0	63.3	66.5

Presented graphically, the impact is quite striking.

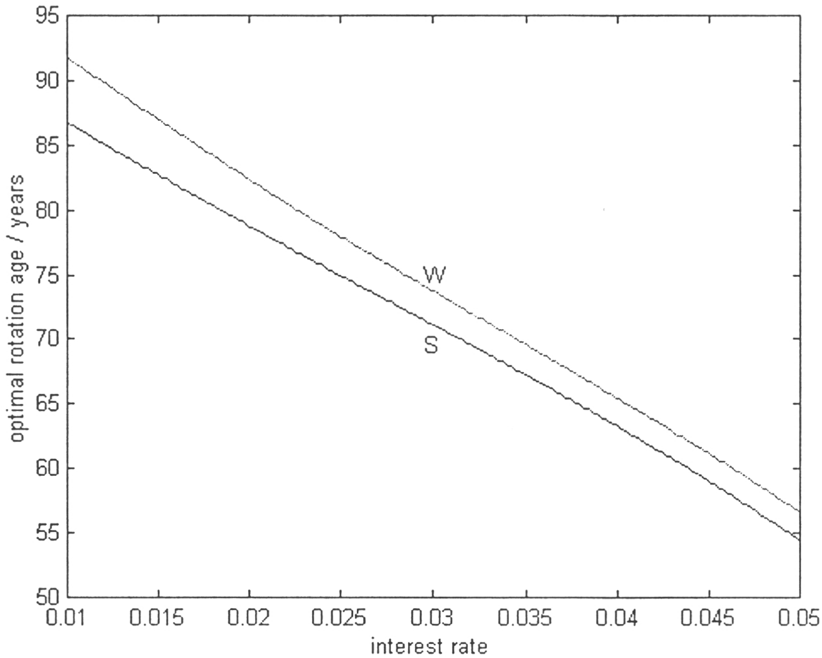

Figure 9. *The optimal rotation with a changing interest rate r in the SEV W and S when p(T) = FIM 240/m³, p'/p = 1%, d(T) = 0.71, d' = 0, C(T) = FIM5000/ha, c'/c = 0.5% and the yield is that in (18) of Vuokila & Väliaho.*

Next, the impact of the rate of change in the stumpage price $p'(\tau)/p(\tau)$ on the optimal rotation period T is both analysed and demonstrated graphically. Let $p(\tau)$ be the exponential, $p(\tau) = p0 \exp(p1\ \tau)$. The partial derivative $\partial T/\partial p1$ at point T is then (see Appendix)

$$\partial T/\partial p1 = - \langle T\, b'(T) + b(T)\{1 - rT / [1-\exp(-rT)]\}\rangle / \langle f''(T)[1-\exp(-rT)] - r^2 f(T)\rangle, \qquad (14)$$

where $b(T) = p(T)q(T)d(T)$ and $f(T) = p(T)q(T)d(T)-c(T)$. Recall that the denominator is negative. Usually both $b(T)$ and $b'(T)$ are positive, as is assumed here. Note that above $\{1-rT/..\} = r^2 T^2/2! - r^3 T^3/3! + \; > 0$, and therefore the comparative statics $\partial T/\partial p1 > 0$ results.

Again using the case in Figure 1 with yield function (18), the graph shows that the price change has a major impact on optimal rotation (Figure 10). More precisely, if the price change p1 increases by 0.1% from zero, the rotation increases by roughly 1.2 years and a 1% increase results in an increase of as much as about 12 years in the rotation (Table 2).

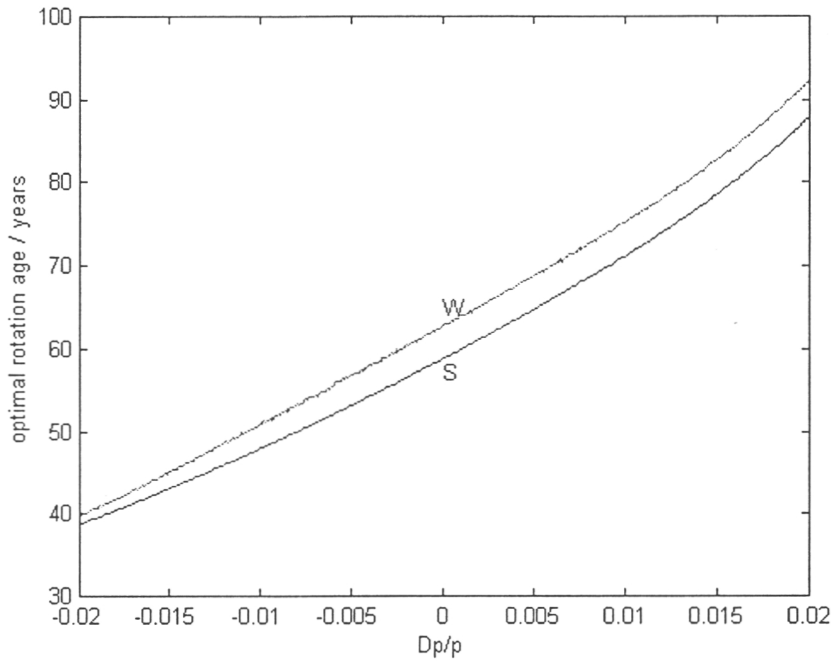

Figure 10. *The optimal rotation with a changing price increase p'/p in the SEV S and W, when r = 3%, p(T) = FIM 240/m^3, d(T) = 0.71, d' = 0, C(T) = FIM5000/ha, c'/c = 0.5% and the yield is that in (18) of Vuokila & Väliaho.*

Table 2. *Sensitivity of S- and W-function with respect to stumpage price change p1.*

Stumpage price Change p1 / %	Optimal rotation time of S-function / years	Optimal rotation time of W-function / years
–1,0	47.9	50.9
–0,9	49.0	52.0
–0,1	57.7	61.4
0.0	58.8	62.8
0.1	60.0	63.9
0.9	69.8	73.9
1.0	71.2	75.3

Table 3. Sensitivity of S- and W-function with respect to reforestation cost change.

Reforestation cost change c1 / %	Optimal rotation time of s-function / years	Optimal rotation time of w-function / years
–2.0	66.3	70.2
–1.5	67.1	70.8
–0.5	69.0	72.1
0.0	70.0	72.9
0.5	71.1	73.7
1.5	73.2	75.1
2.0	73.6	75.0

Next, the impact of the change in the reforestation cost $c'(\tau)/c(\tau)$ is studied. The owner can only react to prices, but he/she can influence reforestation costs. The impact of these costs has been analysed by Chang (1983), among others. He showed that both higher site preparation costs and higher planting costs mean a longer optimal rotation period.

Assume that $c(\tau) = c0 \exp(c1\ \tau)$. The partial derivative of the optimal rotation period at the optimum point T with respect to the change c1 in silvicultural costs $\partial T/\partial c1$ is then (see Appendix)

$$\partial T/\partial c1 = c(T)\ \{1 + (c1 - r)\ T\ /\ [1 - \exp(rT)]\}\ /\ \{\ f''(T) - r^2 f(T)/[1 - \exp(-rT)]\ \}. \quad (15)$$

Recall that the denominator above is negative. Moreover, the numerator is negative and $\partial T/\partial c1 > 0$ as long as $c1 < r$. When $c1 > r$ the sign of $\partial T/\partial c1$ changes and the profitability of forestry is jeopardised (see Figure 11).

Finally, consider the numerical importance of the cost side. The increase in the cost change c1 by 0.1% results in an increase of only 0.2 years in rotation time and a 1% change results in an increase of less than 2 years (Table 3).

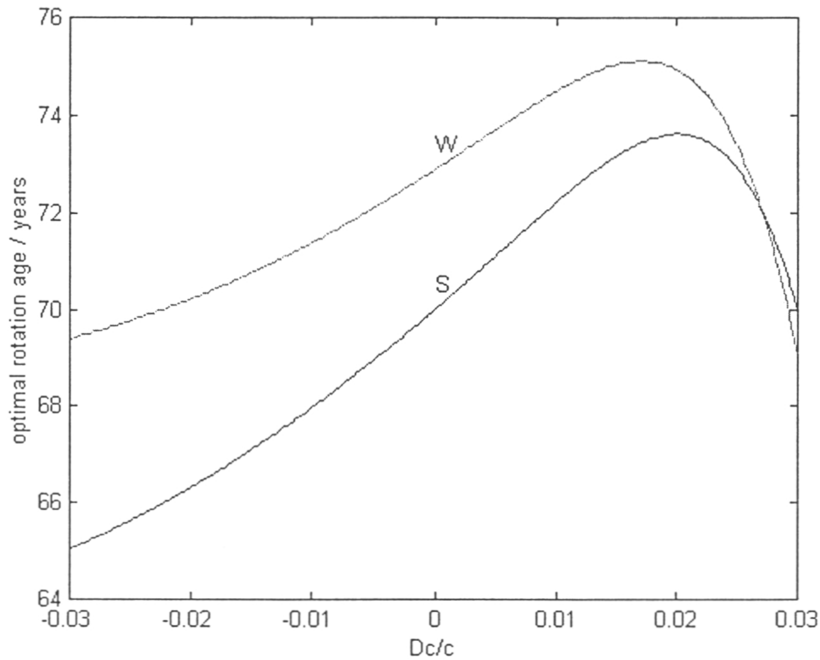

Figure 11. *The optimal rotation with changing cost increase c'/c in the SEV S when r = 3%, p(T) = FIM 240/m^3, p'/p = 1%, d(T) = 0.71, d' = 0, C(T) = FIM5000/h and the yield is that in (18) of Vuokila & Väliaho.*

The graphs above also show that the impact of cost change is quite modest. In the area where the cost increase c1 approaches the interest rate r 3 %, a non-natural situation can been seen as an anomaly in the optimal rotation period.

When the cost increase exceeds roughly 2%, the optimal rotation period decreases dramatically and the profitability of forestry starts to deteriorate. This clearly suggests that a key objective in forest policy should be to keep relative cost increases well below the interest rate level. Recall that the planting costs vary as a function of time, typically tending to increase slightly (Oksanen-Peltola 1989).

3 The numerical solution of the optimal rotation

The optimal rotation solutions (6)–(12) above can be based on growth tables or functions. The solution can then even be constructed manually.[12] Recall that *value* growth is the correct measure related to the sales price, not volume growth. An estimate of the relationship between the value increment and the volume increment is roughly 1.5 according to Nyyssönen & Ojansuu (1982). The annual growth percentage over the next five years in the dominant height, basal area and volume q_{v5} is defined by Vuokila & Väliaho (1980). Yearly volume growth percentage functions have been presented by Nyyssönen & Mielikäinen (1978).

The input variables needed to calculate q_{v5} for a particular stand are available in the forest management planning data. However, rather than use the formulas of Vuokila & Väliaho (1980) with several input variables, three simple volume growth functions are applied here in order to demonstrate the optimal rotation behaviour. The first (I) volume growth function used is simply (Fridh & Nilsson 1980)

$$q_1(\tau) = [\mu \ a \ 1.6416] \, [\, 1. - 6.3582^{(-\tau/a)} \,]^{2.8967}, \tag{16}$$

where μ is the maximum sustainable yield per hectare and a is the age of the stand when μ is obtained (see Lohmander 1987 and Gong 1992).

Note that the standard techniques of numerical analysis are available. In this study, the calculations have been performed using personal computer (PC) and MATLAB software.

Here it is assumed that for a VT (Vaccinium Type) pine stand, which has a dominant height H_{100} of 21 meters at the age of 100 years, one thinning with 35% removal is carried out. The maximum sustainable yield is then 4.1 m^3/ha/y and the maximum sustainable yield age is 100 years (Vuokila & Väliaho 1980, p. 242). Here $\mu = 4$ m^3/ha/y and $a = 100$ years are used.[13]

An alternative volume growth function (II) for testing purposes is that of Kuuluvainen & Tahvonen (1999), which is of the form

$$q_2(\tau) = K / [1 - C \ \exp(-\rho\tau)], \tag{17}$$

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[12] The lower and upper bounds of the correction term $1 / [1-\exp(-rT)]$ above are defined when applying manual calculations. Suppose that the interest rate is, say, 3% per year. Assume initially that the optimal rotation period is between 70 and 90 years. The correction term is then between 1.13954 and 1.07205. This shows that the influence of future generations increases the interest rate r by only about 10%. The key factors are $p'(t)/p(t)$, in addition to the interest rate r. Even the relative cost change $c'(t)/c(t)$ and the relative profit ratio change $d'(t)/d(t)$ have some impact. When these are assumed to be constants, one can use the volume growth tables and the multipliers, which transform volume growth to value growth. When defining the first approximation manually, the reforestation cost ratio $c_r(t)$ can be assumed to be constant. Suppose also that at least good empirical results are available, such as growth and yield tables (e.g. Vuokila & Väliaho 1980).

[13] The average sustained yield for a Vaccinium Type (VT) site in Southern Finland is at most 4.7 m^3/ha/y and, according to the Central Forestry Board taxation tables, 4.0 m^3/ha/y.

where $K = q_2(\infty) = 500\ m^3$, $C = (q0-K)/q0$, $q0 = q_2(0) = 10\ m^3$ and a growth rate $\rho = 0.048$ are the parameters proposed by the authors.

Third, consider a traditional and well known differential equation $q'(\tau)/q(\tau) = [q(\tau)-q_b][q_\infty - q(\tau)]\ g(\tau)$ (see e.g. Hald 1952, p. 659), where the growth is related to the distance from both the bottom q_b and the ceiling q_∞. The solution $q(\tau)$ of the differential equation above is affected by $g(\tau)$, a special function of τ. However, the limitations in the amount of observations suggest that function $g(\tau)$ is constant. Then the solution of the differential equation including a nonzero bottom q_b gives the third (III) volume function

$$q_3(\tau) = q_\infty / [1 + C \exp(-\tau/a)] + q_b, \tag{18}$$

where $C = [q_\infty / q_0] - 1$, $q_\infty = q_3(\infty) - q_b$, $q_0 = q_3(0) - q_b$ and age a is the shape parameter.

This is applied to curve fitting for the calculations after the last thinning (Vuokila & Väliaho 1980, p. 242). Total production for the dominant height h 24 m is $450\ m^3$ (Hynynen & Ojansuu 1996, p. 73) and for h 21 m roughly $420\ m^3$. The thinning means a removal of $65\ m^3$. Then the estimates for q_∞ and q_b are $355\ m^3$ and $-65\ m^3$.

The estimation based on the calculations of Vuokila & Väliaho (1980, p. 242), suggests, however, $460\ m^3$ for total production and $-85\ m^3$ for the bottom q_b. The starting volume q_0 estimate is $22.8\ m^3$ and the shape parameter a estimate 19.5 years, or ρ as in Kuuluvainen & Tahvonen (1999), 0.051 $year^{-1}$. All the growth and yield table figures q and respective estimates $q(\tau)$ were compared, and the maximum deviation turned out to be only 4.2%.

4 Applications

Consider first a comparison of a manually obtained optimal rotation result and Tapio's recommendations. Suppose that the annual interest rate for the whole economy is $r = 3\%$, that the annual price increase is the same as the local trend for this century $p'(T)/p(T) = +0.4\%$ and that the annual change in the gross profit ratio, say, $d'(T)/d(T) = -0.1\%$. Assume, for the phase of manual calculations, that the reforestation ratio $c_r(T)$ is constant, $c_r(T) = c(T)/[p(T)q(T)d(T)] = 10\%$. Moreover, the correction term $1 / [1 - \exp(-rT)]$ is between 1.07205 and 1.13954 for $r = 3\%$/year, whenever $70 < T < 90$ years. The optimal rotation of $S(t)$ in (6) and (7) is then defined by the value increase $q'(T)/q(T) = 1.9\%–2.1\%$ per year. Given these percentages, the volume growth tables (Vuokila & Väliaho 1980, p. 220) for pine on Vaccinium type (VT) (height index $H_{100} = 24$) sites with three thinnings suggest an optimal rotation period of roughly 85 years. The optimal rotation recommended by Forestry Centre Tapio (1994) for a dryish upland forest site VT under pine is 90–100 years.

Second, suppose that the increase in yearly regeneration costs is 0.5% – 1% per year, i.e. $c'(T)/c(T)= 0.5\%$ per year. The cost of planting is $c_a = 5000$ FIM/ha and that of natural reforestation is $c_n = 3000$ FIM/ha, i.e. $c(T) = 5000$ or 3000 FIM/ha. Additionally, assume that the standard present list price for pine logs $p(T)$ is FIM240/m^3 and that the gross profit ratio is $d(T) = 71\%$, after sales tax of 29%. The volume increase is multiplied by 1.5 to obtain the value increase (Nyyssönen & Ojansuu 1982), which is recognised by the linear price change between 40 and 80 years. Now, when using growth formula (16) with the parameters $\mu = 5.0$ m^3/ha/y and $a = 80$ years, i.e. the age of the stand when μ is obtained, numerical methods give an optimal rotation of $T = 73$ years for planting and 71 years for natural regeneration. With yield function (17) the corresponding optimal rotation periods are $T = 78$ years for natural regeneration and $T = 81$ years for planting. The natural generation option involves a transition at the beginning of the growth period of, say, 5–10 years, implying that the actual rotation period recommendation is nearly 80 years.

Third, the inaccurate value growth ratio of 1.5 (Nyyssönen & Ojansuu 1982) can be eliminated by applying directly the log and pulpwood volumes of Vuokila & Väliaho (1980, p. 242). The roundwood prices are FIM271 / m^3 for pine log and FIM87 / m^3 for pine pulpwood (Aarne & Linna 1999). As before in (18) a logistic growth function (see e.g. Hald 1952, p. 659) is applied. The solution of gross income $g(t)$, price times quantity, based on Vuokila & Väliaho (1980, p. 242), including also a nonzero bottom g_b, is

$$g(t) = g_\infty / [\, 1 + D \exp(-t/s)] + g_b, \tag{19}$$

where $g_\infty = g(\infty) - g_b =$ FIM95,338, $D = [(g_\infty)/ g_0] - 1 = 106.0225$, $g_b = g_0 - g(0) =$

–FIM 7,473.4 and age $s = 15.1507$ is the shape parameter providing $1/s = 6.6\%$/year.

Now the gross income function (19) above is used for analysing the two period model W(t) of (8), which uses "dynamic" parameters for period n and the model S(T) of (6) for periods, n+1, n+2, ...with "trend" parameters. The optimal rotation is then determined using the case considered in Figure 1 with varying silvicultural costs. The other parameters are of the same order as in Figures 1 and 2 (see Figure 12).

Surprisingly, the impact of reforestation costs turned out to be modest –only some two years – even when the gross income growth model was used.

Next, the impact of the forest owner's interest rate r on the optimal rotation age is analysed.

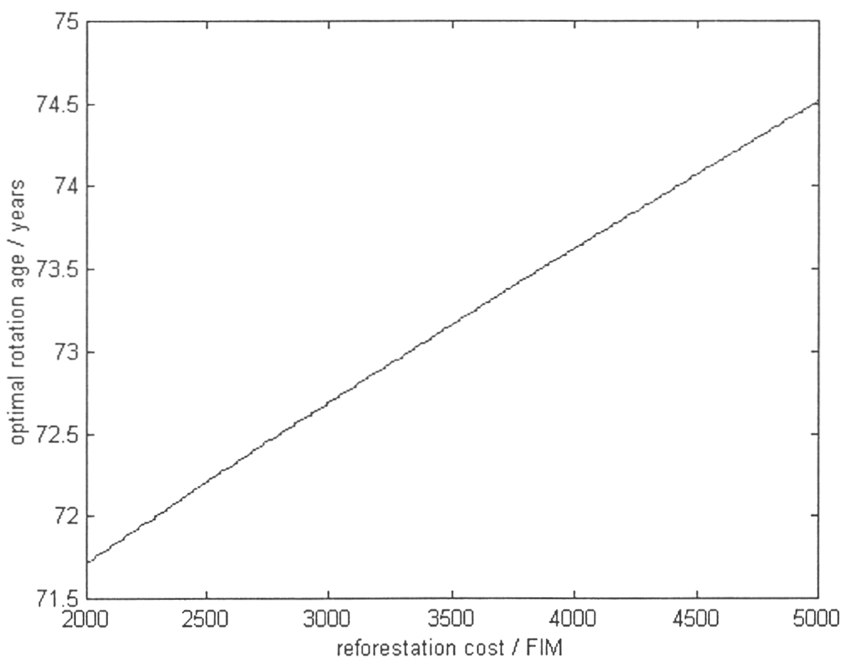

Figure 12. *The optimal rotation period with soil expectation value W in (8) varying present reforestation costs, when c'/c = 0.5%, p(pine pulpwood) = FIM86, p(pine log) = FIM 271, r = 3%, d(T) = .71, d' =0 and gross income is that in (19).*

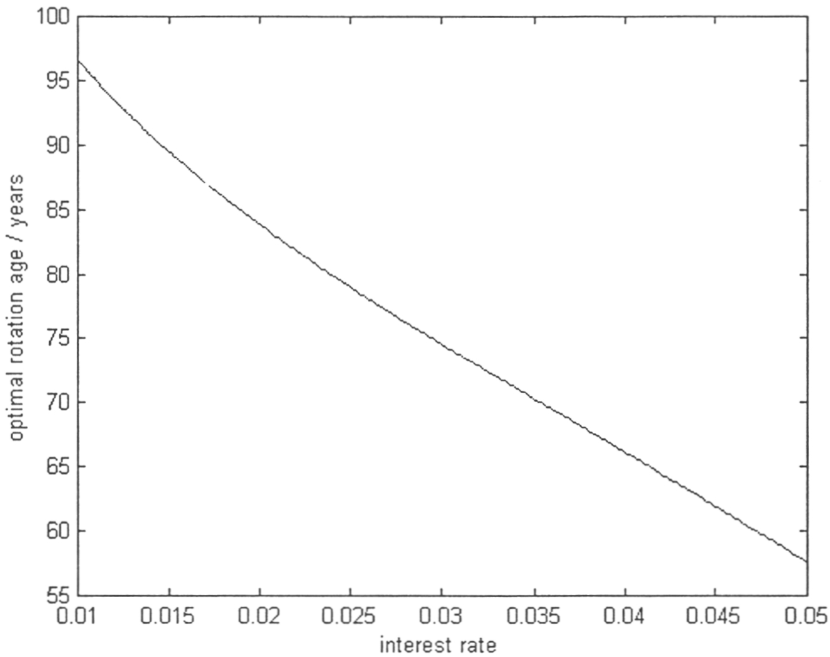

Figure 13. *The optimal rotation period with soil expectation value W in (8) varying interest rate r, when c(T) = FIM5000, c'/c = 0.5%, p(pine pulpwood) = FIM86, p(pine log) = FIM 271, d(T) = .71, d' =0 and gross income is that in (19).*

As expected, the impact is dramatic. If a forest owner has loans on which interest is payable at an annual rate of, say, 5–6%, he might be interested in ignoring not only the recommendations of Forestry Center Tapio (1994) but also the legal provisions concerning the lower limit of the final felling age (Ministry of Agriculture and Forestry 1997).

Finally, the optimal rotation period is analysed as a function of the change in the present log price p0. The pulp price is assumed constant at FIM86/m^3. Both soil expectation value (SEV) functions S(t) in (6) and W(t) in (8) are applied.

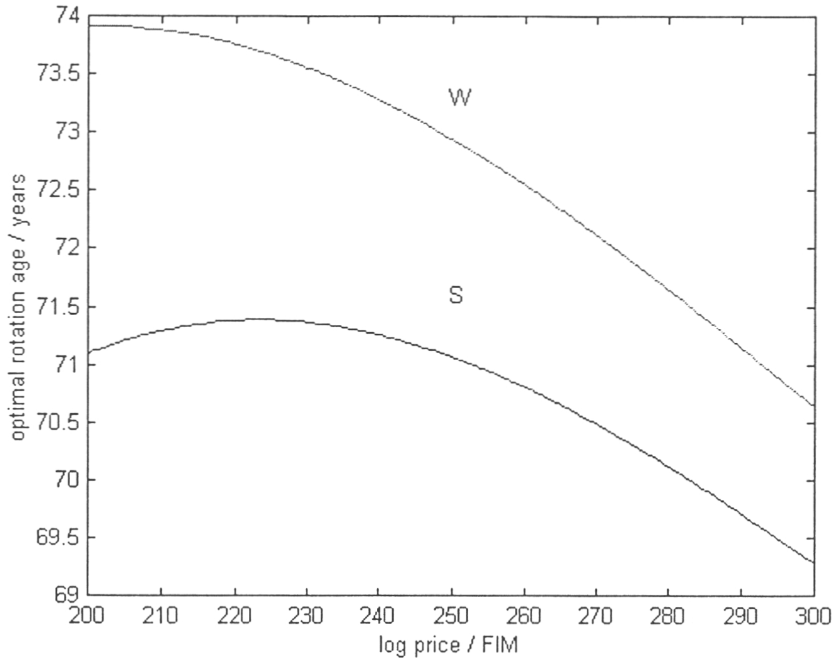

***Figure 14**. The optimal rotation period with soil expectation value W in (8) varying log price p0, when c(T) = FIM5000, c'/c = 0.5%, p(pine pulpwood) = FIM86, r = 3%, d(T) = .71, d' =0 and the gross income is that in (19).*

The level remained well below 80 years, which is the lower limit required by law, even with the inclusion of the gross income growth model and other improvements and "realistic" features of the modelling.

It must be emphasised that stochastic (random) stumpage prices might increase the optimum rotation age slightly. The inclusion of decreasing relative harvesting costs with age may also increase the optimal solution.

All in all, the 90 years recommendation of Forestry Centre Tapio (1994) is cautious. The lower limit set for regeneration felling, 80 years in Southern and Central Finland, by the Ministry of Agriculture and Forestry (1997) roughly corresponds to the final results obtained with an interest rate of 2.5%.

Discussion

This study has analysed the optimal rotation problem when economic factors such as stumpage prices, reforestation costs, interest rates and gross profit ratios are allowed to vary in time according to both biological age and calendar time.

The methodological problem was tackled by applying dynamic programming (DP). The existence and uniqueness of global optimal rotation periods was based on DP, the induction axiom and the explicit quasi-concavity of the objective functions, which are first non-decreasing and then non-increasing. Forest economics has traditionally relied on concavity, which has thus been essentially relaxed here.

The bare land value inspires two different cases: (i) future prices and costs depend on age and (ii) also on calendar time. All the combinations were investigated applying the same and different functions for both the present and future periods. However, when the prices and costs of future rotations also recognise calendar time, they depend only exponentially on calendar time because of the limited availability and accuracy of forecasted price and cost estimates.

Different volume growth models and a value growth model for pine on a Vaccinium (VT) site type were applied in analysing the optimal rotation and its sensitivity. Models based on Vuokila & Väliaho were, however, found to be the most practicable for the study. It turned out that planting regeneration results lead to optimal rotations that are approximately only three years longer than those for natural regeneration, to which the delay at the inception of the growth should be added. Analogously, an increase in cost change velocity by one per cent affects the rotation y by roughly two years only. However, a one per cent increase in both the velocity of price change and the interest rate produces a jump of some 10 years in the rotation. All the models highlight the sensitivity of the optimal rotation to the price change. The income growth model produces a rotation period that is approximately some five years longer than that of the corresponding volume growth model.

Price change has a fundamental impact on the length of the optimal rotation period, which suggests that forest owners should sell when prices are at their peak. Rotation lengths are also heavily influenced by the interest rate prevailing in the economy, which is taken a given. However, individual forest owners face different personal interest rates, because of loans etc. The impact of the reforestation cost is negligible. The results reveal strong dynamics produced by the current market situation. Surprisingly, when the cost increase approaches the interest rate, the optimal rotation period starts to decrease dramatically and the profitability of forestry deteriorates.

The key issue is the availability of unbiased parameter estimates, as well as accurate value and volume growth models for different tree species and forest stands. A striking outcome was the impact of relative value growth on the optimum. The 90 years recommended by Forestry Centre Tapio (1994) turned out to be cautious. The legally required rotation age limit of 80 years, as interpreted by Ministry of Agriculture and Forestry (1997), was roughly similar to the effect of an interest rate of 2.5%.

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Appendix

The traditional approach applied in optimal rotation studies, and even more generally in forest economics, has relied on *concavity*, i.e. proved that the second order derivative of the soil expectation value (SEV) with respect to rotation age t is negative in order to guarantee a global unique maximum (see e.g. Chang 1983, 1984). Unfortunately, concavity does not necessarily hold (Figure 15).

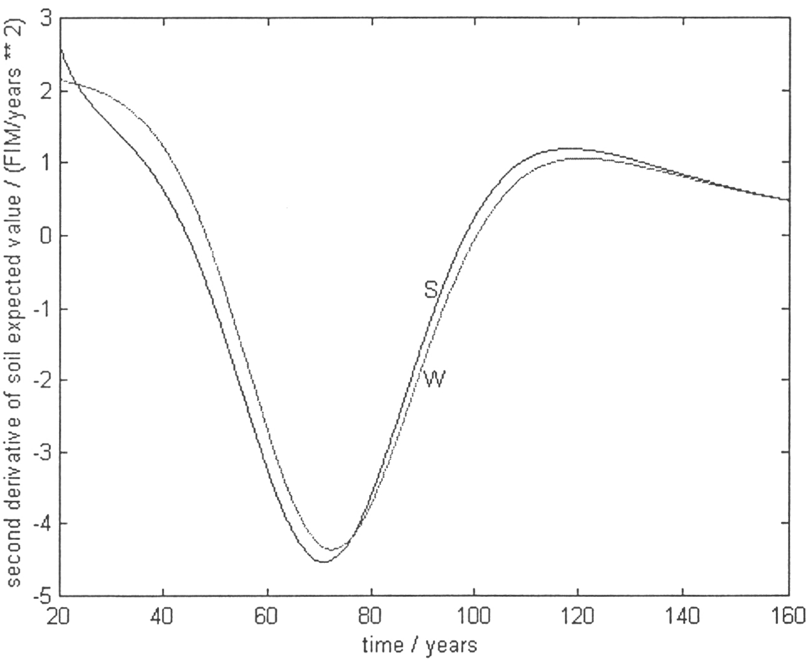

Figure 15*. The second derivative soil expectation values S and W, when r = 3%, $\pi(T)$ = FIM 24/m^3, $\pi'/\pi = p'/p$ = 1%, $\delta(T)$ = 0.71, $\delta'/\delta = d'/d$ = –0.05%, c(T) = FIM5000/ha, $\gamma'/\gamma = c'/c$= 0.5% and the yield is that in (18) of Vuokila & Väliaho.*

However, it was shown already by Martos (1965) that a more general class of *explicitly quasi-concave*[14] functions is a *sufficient* condition, which guarantees that any local maximum is also a global maximum.

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[14] A real–valued function f defined on a convex subset E of R^n is (Danao 1992):
- quasi-concave on E if and only if $x,y \in E$, $\lambda \in [0,1]$, and $f(x) \le f(y)$ imply $f(x) \le f[(1-\lambda)x+\lambda]$;
- semistrictly quasi-concave on E if and only if $x,y \in E$, $x \neq y$, $\lambda \in (0,1)$, and $f(x) < f(y)$ imply $f(x) < f[(1-\lambda)x+\lambda y]$;
- explicitly quasi-concave on E if and only if it is quasi-concave and semistrictly quasi-concave on E.

Recall that a local maximum of a quasi-concave function f is a global maximum or f is constant in the neighbourhood of the local maximum (Greenberg & Pierskalla 1971). Note that the quasi-concavity of f is equivalent to the quasi-convexity of $-f$ and vice versa (Martos 1975). In the absence of constraints and given a continuous function f defined over a convex set, explicit quasi-concavity is also *necessary* for any local maximum to also be a global maximum (Netzer & Passy 1975). Recently, the sufficient property to guarantee that a local maximum is also a global one has been relaxed to semistrictly quasi-concavity where semicontinuous functions are concerned (Daniilidis & Hadjisavvas 1999). Moreover, if a *strictly quasi-concave*[15] function has a maximiser, then it is unique (Danao 1992).

Recall the key notion that for an *explicitly quasi-concave* function from a convex set in R^n any local maximum is global (see Martos 1975, p. 89). Note also that a product of concave nonnegative functions is explicitly quasi-concave. Even a concave non-negative function divided by a convex positive function gives an explicitly quasi-concave function (Martos 1975, p. 61–63).

Note that in order to avoid anomalies all functions are here assumed to be continuous and differentiable and to possess derivatives of first and second order.

The derivation of the soil expectation value $V(\tau)$ in (4) is given by modifying a finding of Nautiyal & Williams (1990) as follows: instead of $p\,q(x,\tau)$, $p(\tau)q(\tau)d(\tau)$ is used. Denote $p(\tau)q(\tau)d(\tau) = b(\tau)$. The derivative $V'(\tau)$ is then simply

$$V'(\tau) = \{ [b'(\tau)-r\,b(\tau)]/[\exp(r\tau)-1] \} - r\,[b(\tau) - c(0)\exp(r\tau)] \} / \{[\exp(r\tau)-1]^2\}.$$

Now $V'(T) = 0$ implies

$$b'(T) - r\,b(T) = r[b(T)-c(0)]\exp(rT)/[\exp(rT)-1] \quad \text{and}$$

$$b'(T)[\exp(rTt)-1] - r\,b(Tt)\exp(rT) = -c(0)\exp(rTt)$$, which gives (5).

Both the numerator $b(\tau)\exp(-r\tau) - c(0)$ and denominator $1-\exp(-r\tau)$ of $V(\tau)$ in (4) are differentiable, the denominator even being convex and > 0 for $\tau > 0$. $V(\tau)$ is explicitly quasi-concave, whenever the numerator is non-negative and concave (Martos 1975, p. 63). The numerator is concave, whenever its derivative is non-increasing. The non-negativity holds after some $\tau_0 > 0$. Then $V(\tau)$ is explicitly quasi-concave after some $\tau_0 > 0$. Then any local maximum $V(\tau)$ after some $\tau_0 > 0$ is global.

The optimal rotation solution (7) is based on $S(\tau)$ in (6), the derivative of which

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[15] A real-valued function f defined on a convex subset E of R^n is (Danao 1992):

– strictly quasi-concave on E if and only if $x,y \in E$, $x \neq y$, $\lambda \in (0,1)$, and $f(x) \leq f(y)$ imply $f(x) < f[(1-\lambda)x + \lambda y]$.

Note that a strictly quasi-concave function is explicitly quasi-concave according to Danao (1992). However, some earlier sources such as Greenberg & Pierskalla (1971) defined strictly quasi-convexity in such a way that it did not even imply quasi-convexity. With lower semicontinuous functions the two definitions, explicit quasiconvexity and strict quasi-conexity coincide (Greenberg & Pierskalla 1971). The same holds with quasi-concave and upper semicontinuous functions, because any convexity property of f is equivalent to the respective concavity property of $-f$.

is analogous when using the notation $b(\tau)$

$$S'(\tau) = \{[b'(\tau)-c'(\tau)] / [\exp(r\tau)-1]\} - \{r[b(\tau)-c(\tau)]\exp(r\tau)\} / \{\exp(r\tau)-1\}^2.$$

Letting $S'(T) = 0$ and multiplying it by $[(\exp(rT)-1]$ and dividing by $b(T)$ and denoting $c(T)/[p(T)q(T)d(T)]$ by $c_r(T)$, one obtains (7).

Ignore $c(0)$ for a while. The denominator $[\exp(r\tau)-1]$ of $S(\tau)$ is convex and > 0 for $\tau > 0$. Whenever the numerator $b(\tau)-c(\tau)$ is concave and non-negative, $S(\tau)$ is explicitly quasi-concave (Martos 1975, p. 63). The concavity of the numerator is implied whenever the derivative of $b(\tau)-c(\tau)$ is non-increasing.

Consider $W(t)$ in (8), ignoring the constant $c(0)$. All terms are divided by $\exp(rt)$, which is both positive and convex. Recall that the sum of concave functions is concave. The constant $S(T_{n+1})$ is trivially concave. Provided that the term $[\pi(t)q(t)\delta(t)-\gamma(t)]$ is concave, which is the case when its derivative is non-increasing or the second derivative is negative, then $W(t)$ is explicitly quasi-concave.
Consider

$$Z(t) = c(0) + b(t)\exp(-\alpha t) - c(t)\exp(-\beta t) + b(T_{n+1})\exp(-\alpha(t+T_{n+1})) - c(T_{n+1})\exp(-\beta(t+T_{n+1}) + b(T_{n+2})\exp(-\alpha(t+T_{n+1}+T_{n+2})) - c(T_{n+2})\exp(-\beta(t+T_{n+1}+T_{n+2}) + \dots .$$

ignoring $c(0)$ in the context of DP. The contributions of each period are separable, and each period is connected to the future with decreasing exponential multipliers. The DP type of problem definition is,

$$Z_n(t) = \max_t \{ b(t)\exp(-\alpha t) - c(t)\exp(-\beta t) + \exp(-\alpha t) Z_{+,n+1}(T_{n+1}) - \exp(-\beta t) Z_{-,n+1}(T_{n+1}) \},$$

where $Z_{+,n+1}(T_{n+1})$ stands for the benefit components and $Z_{-,n+1}(T_{n+1})$ the cost components of the future. The formulation and the functions are exactly the same for each period $n, n+1, n+2, \dots$. Since the formulation holds for n, is assumed to hold for $n+k$, for any $k > 0$ and then holds for $n+k+1$ as shown above, the induction axiom implies that $T = T_n = T_{n+1} = T_{n+2} = T_{n+3} = \dots$. When the periods are equal (10) reduces to a kind of steady state form $Z_s(t) = c(0) + b(t)/[\exp(\alpha t)-1] - c(t)/[\exp(\beta t)-1]$, which gives (10').

Recall that $\alpha, \beta > 0$ is required in order to avoid the explosion of the model. Then both denominators $\exp(\alpha t)$ and $\exp(\beta t)$ above are non-negative and convex. Whenever $b(t)$ is concave and $c(t)$ is convex both items above are explicitly quasi-concave. If their sum $b(t)-c(t)$ is concave, explicitly quasi-convex hull functions of $Z_s(t)$ can be constructed by replacing first α by β and then β by α in (10'). In practice, simple functions of $p(\tau)$, $d(\tau)$ and $c(\tau)$ such as exponential and/or linear ones are sufficient for the calculations. Then the explicit quasi-concavity or even concavity of $Z_n(t)$ itself can be based on the concavity of its components.

Consider the existence and uniqueness of solution (12) of

$$W_z(t) = c(0) + \varphi(t)\exp(-\alpha t) - \gamma(t)\exp(-\beta t) + \exp(-\alpha t)\, b(T_{n+1})/[\exp(\alpha T_{n+1})-1] - \exp(-\beta t)c(T_{n+1})/[\exp(\beta T_{n+1})-1].$$

The terms multiplied by $\exp(-\alpha t)$, or divided by convex and positive $\exp(\alpha t)$, are explicitly quasi-concave whenever $\varphi(t)$ is concave. In the same way, the terms multiplied by $\exp(-\beta t)$ are explicitly quasi-concave whenever $\gamma(t)$ is convex. Then explicitly quasi-concave hull functions can be constructed as above by replacing first α by β and then β by α in $W_z(t)$.). Recall that, in practice, simple functions of $p(\tau)$, $d(\tau)$ and $c(\tau)$ such as exponential and/or linear functions are typically used in the calculations. Then the explicit quasi-concavity of the function $W_z(t)$ itself can be based on the concavity of its components.

The sensitivity of the solutions in the optimal rotation point T in the case of S (T) is considered. The derivative $S'(T) = \{f'(T) - r\, f(T)/[1-\exp(-rT)]\}/\{\exp(rT)-1\} = 0$, where $f(T)=p(T)q(T)d(T)-c(T)$, consists of two terms the first of which = 0 and the last of which > 0. The differentiation of the implicit function $S'(T)= 0$ gives $\partial T/\partial r = -[\partial S'(t)/\partial r] / [\partial S'(t)/\partial T]$. Recall that $\partial S'(T)/\partial T = S''(T)$ is, when using the optimum points condition $f'(T) - r\, f(T)/[1-\exp(-rT)]= 0$, simply

$$S''(T) = \{ f''(T) - r^2 f(T)/[1-\exp(-rT)] \} / \{\exp(rT)-1\}.$$

When using the above conditions at the optimum point T, the derivative $\partial S'(T)/\partial r$ is

$$\partial S'(T)/\partial r = -\{f(T)/[1-\exp(-rT)\} \{1 - rT / [\exp(rT)-1]\} / \{\exp(rT)-1\}.$$

The implicit derivative $\partial T/\partial r$ at the optimum point T is then

$$\partial T/\partial r = \{ f(T) [1 - rT / [\exp(-rT)-1] \} / \{f''(T)[1-\exp(-rT)] - r^2 f(T) \},$$

which gives (13).

The price function is now assumed to possess the form $p(t) = p0 \exp(p1\, t)$.

Then $\partial S'(T)/\partial p1$ is, denoting $b(T) = p(T)q(T)d(T)$, at the optimum point T

$$\partial S'(T)/\partial p1 = < T\, b'(T) + b(T)\{1 - rT / [1-\exp(-rT)]\} > / < \exp(rT)-1> .$$

The sensitivity of T with respect to p1 is thus

$$\partial T/\partial p1 = -<T\, b'(T) + b(T)\{1- rT / [1-\exp(-rT)]\}> / <f''(T)[1-\exp(-rT)] - r^2 f(T)>,$$

which gives (14).

If the cost function has the form $c(t) = c0 \exp(c1\, t)$, the derivative $\partial S'(T)/\partial c1$ is

$$\partial S'(T)/\partial c1 = -c(T) \{1 + c1\, T - rT/[1-\exp(-rT)]\} / \{\exp(rT)-1]\}.$$

Finally, the sensitivity of T with respect to c1 is thus

$$\partial T/\partial c1 = c(T) \{1 + c1\, T - r\, T / [1-\exp(rT)]\} / \{ f''(T) - r^2 f(T)/[1-\exp(-rT)] \},$$

which yields (15).

ISBN 951-40-1751-X
ISSN 0358-4283
Hakapaino Oy 2000